IFT-Perimeter-SAIFR
Journeys into Theoretical Physics 2024
Sunday Exam

- Write your name on each page

- Number each page used to solve a given problem as $1/n, 2/n, \ldots, n/n$ where n is the number of pages used to solve that problem

- Do not solve more than one problem per page – these exams will be split apart and graded by different people.

- Problem 1 (Cosmology): 25%
- Problem 2 (Quantum Matter and Mean Field): 25%
- Problem 3 (Quantum Information): 25%
- Problem 4 (4 Observers in AdS): 25%

◦ Full Name: ______________________________

◦ I am interested in applying to the IFT masters program even if I am not accepted into the PSI program: [Yes] [No]

◦ If accepted into any of the programs, I would be interested in starting my fellowship at the IFT in [August 2024] [March 2025] [August 2025] [March 2026] (choose one)

◦ The areas of physics which I am most interested in are:

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Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 Cosmology

1.1 Some formulas

FLRW metric:

$$ds^2 = -c^2 dt^2 + a^2(t)\, d\vec{l}^2 \quad , \quad d\vec{l}^2 = \gamma_{ij}\, dx^i dx^j = d\chi^2 + \frac{1}{K} \mathrm{sen}^2 \left(\sqrt{K} \chi \right) d\Omega^2$$

Comoving and physical distances:

$$\Delta l_c = \int dl \quad , \quad \Delta l_f = a(t)\, \Delta l_c$$

Cosmological redshift:

$$1 + z = \frac{1}{a} \quad , \quad \nu_{obs} = \frac{\nu_{em}}{1+z} \quad , \quad \lambda_{obs} = \lambda_{em}\,(1+z)$$

Friedmann equations:

$$G_{00} = 8\pi G T_{00} \quad \Rightarrow \quad 3\frac{\dot{a}^2}{a^2} + 3\frac{K}{a^2} = 8\pi G\, \rho$$

$$G_{ij} = 8\pi G\, T_{ij} \quad \Rightarrow \quad -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{K}{a^2} = 8\pi G\, p$$

Continuity equation for a cosmological fluid:

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0 \quad , \quad H = \frac{\dot{a}}{a}$$

Friedmann equations for the fractions (Ω_i) of density in cold matter ("dust"), radiation, vacuum energy ("cosmological constant" Λ) and spatial curvature:

$$H^2(z) = H_0^2 \left[\Omega_m (1+z)^4 + \Omega_m (1+z)^3 + \Omega_k (1+z)^2 + \Omega_\Lambda \right]$$

1.2 Question

Let's consider an universe that behaves in the following way:

(I) $\rho = \rho_I$ =constant for $z \geq 99$, an initial era dominated by vacuum energy;

(II) $\rho = \rho_{II}\,(1+z)^3$ for $1 \leq z \leq 99$, an intermediate era dominated by dust; and

(III) $\rho = \rho_{III}$ =constant for $z \leq 1$, a late epoch dominated by another type of vacuum energy ("dark energy").

We will assume here that spatial curvature is negligible, $K = 0$. Clearly, for continuity we must have $\rho_I = 10^6\, \rho_{II}$, and also that $\rho_{III} = 8\, \rho_{II}$.

Let's make a radical approximation and take the Hubble parameter to be expressed in those three epochs as simply:

(I) $H(z) = H_I$

(II) $H(z) = H_{II}(1+z)^{3/2}$

(III) $H(z) = H_{III} = H_0$

1. [**5pt**] Assuming that we measure the expansion rate today and get the value H_0 , compute the values of the densities of the three components, ρ_i, as well as H_I and H_{II} .

2. [**10pt**] Calculate the age of the universe in this scenario. However, here you will have to assume that the initial era dominated by vacuum energy (I) only exists up to some maximum redshift z_{max}. Compute the age for $z_{max} = 10^4$ and $z_{max} = 10^{10}$. You should leave your answers in terms of H_0 .

3. [**10pt**] Compute the light cone (past and future) for an event at $\chi = 0$ and $t = t_0$ (the time at $z = 0$) . Make a diagram for this light cone (this is a plot of $t \times \chi$ corresponding to the light cone), and if you notice that there are past or future horizons, write their values.

2 Quantum Matter, Mean Field Theory

Consider the 1d transverse field Ising model with Hamiltonian

$$H = -J \sum_{i=1}^{L} Z_i Z_{i+1} - g \sum_{i=1}^{L} X_i,$$

and periodic boundary conditions, $Z_{L+1} = Z_1$.

For $g = 0$, the ground states can be chosen to be two product states polarized in Z direction, whereas for $J = 0$, the ground state is polarized in X direction.

1. [**10pt**] Write down a product state wavefunction with a variational parameter that can interpolate between the two extremes above.

2. [**10pt**] For such a variational product state, find the variational parameter which minimizes the expected value of the energy with respect to the Hamiltonian H, as a function of g/J.

3. [**5pt**] Sketch how $\langle Z \rangle$ varies with g/J, for the optimized variational wavefunction. Interpret the results.

3 Quantum Information

Another magic property of entanglement

Let us consider the so-called *Mermin's game*. Suppose that we have three players, Alice, Bob, and Charlie. Suppose each receives one bit of information: Alice receives x, Bob gets y, and Charlie receives z. They do not know what the others have received but know that $x + y + z = 0 \mod 2$.

Their goal is to produce three output bits (one for each and denoted by a, b, and c, respectively) such that the following constraint is verified

$$\begin{aligned} a + b + c = 0 \quad \mod 2 \qquad & \text{if } x = y = z\,, \\ a + b + c = 1 \quad \mod 2 \qquad & \text{elsewhere}\,. \end{aligned}$$

1. **[5pt]** Show that they cannot win with certainty. Describe a strategy allowing them to win with probability $p = 3/4$.
2. Let us now assume that they have the following entangled 3-qubit state, in such a way that each of them possesses one qubit

$$|\psi\rangle \equiv \frac{1}{2}\Big(|000\rangle - |011\rangle - |101\rangle - |110\rangle\Big)\,.$$

Suppose that each player does the following: if the bit in their possession is 1, they apply a Hadamard, otherwise, they do nothing.

   (a) **[10pt]** Compute the resulting state.

   (b) **[10pt]** Use it to deduce a strategy that wins with certainty.

4 Four Observers in AdS

Consider AdS_3 in global coordinates

$$ds^2/R^2 = -\cosh^2(\rho)dt^2 + d\rho^2 + \sinh^2(\rho)d\phi^2 \tag{1}$$

A and B are two observers, both stationed at fixed AdS coordinate $\rho = \rho_0$ and separated from each other as much as possible. C is a third observer following a geodesic circular orbit centered around $\rho = 0$ and passing by these two points. Finally there is a static observer D sitting in the center of AdS at $\rho = 0$.

1. [**4pt**] Draw the four observers in the global AdS cylinder.
2. [**2pt**] A and B are always at the same proper distance from each other for any global time t. What is this distance as a function of ρ_0?
3. [**2pt**] What is the proper length of the perimeter of a fixed t circle centered at $\rho = 0$ and passing by $\rho = \rho_0$?
4. [**2pt**] What is the proper area of the disk $\rho \leq \rho_0$ enclosed by that circle?
5. [**2pt**] Are A and B following geodesics? Explain. What about D? Explain.
6. [**4pt**] How long does it take – according to D – for C to go around a full orbit?
7. [**4pt**] How long does it take – according to A – for C to go around a full orbit?
8. [**5pt**] At time t_1 observers A and B – who are at rest w.r.t. observer D – start falling freely towards the centre. Eventually they both collide with D at the origin. At what time t_2 does that happen? What velocity does A have according to D at that collision point? What velocity does A have according to B at that same point?

You might find useful to recall the formulae

$$1 = R^2(\cosh^2\rho\,\dot{t}^2 - \dot{\rho}^2 - \sinh^2\rho\,\dot{\phi}^2) \quad \text{proper velocity is normalized} \tag{2}$$

$$E = R^2\cosh^2\rho\,\dot{t} \quad \text{energy conservation from } t \text{ translation invariance} \tag{3}$$

$$J = R^2\sinh^2\rho\,\dot{\phi} \quad \text{angular mom. cons. from } \phi \text{ translation invariance} \tag{4}$$

which we derived in class for the particle action

$$L = \frac{R^2}{2}\int ds(-\cosh^2\rho\,\dot{t}^2 + \dot{\rho}^2 + \sinh^2\rho\,\dot{\phi}^2)\,. \tag{5}$$