IFT-Perimeter-SAIFR
Journeys into Theoretical Physics 2024
Saturday Exam

- Write your name on each page

- Number each page used to solve a given problem as $1/n, 2/n, \ldots, n/n$ where n is the number of pages used to solve that problem

- Do not solve more than one problem per page – these exams will be split apart and graded by different people.

- Problem 1 (Equilibrium and Hermite Polynomials): 25%
- Problem 2 (Localization in 1D via the Landauer approach): 25%
- Problem 3 (Lengths and Spaceship): 25%
- Problem 4 (Aharanov-Bohm Effect): 25%

◦ Full Name: ____

◦ I am interested in applying to the IFT masters program even if I am not accepted into the PSI program: [Yes] [No]

◦ If accepted into any of the programs, I would be interested in starting my fellowship at the IFT in [August 2024] [March 2025] [August 2025] [March 2026] (choose one)

◦ The areas of physics which I am most interested in are:

____

Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 Equilibrium and Hermite Polynomials

In this problem we have N particles in an infinite line at positions $x_1 < x_2 < \cdots < x_N$. Particle i feels an external force $F(x_i) = -x_i$ as well as an interaction force $f(x_i - x_j)$ for the interaction with each of the other particles. The location of the particles at equilibrium is thus given by the vanishing of the sum of all forces,

$$F(x_i) + \sum_{j \neq i} f(x_i - x_j) = 0\,, \qquad i = 1, \ldots, N\,. \tag{1}$$

Note that for x positive (negative) the force $F(x)$ is negative (positive) so that the external force tries to confine the particles around the origin. This problem has two parts A and B for two different interaction forces $f(x)$:

$$\text{Problem A:} \qquad f(x) = \frac{1}{x} \qquad , \qquad \text{Problem B:} \qquad f(x) = \frac{2}{x^3} \tag{2}$$

We will establish the following remarkable theorem:

The particle's locations x_i which are determined by (1) are the same for both problems and are nothing but the roots of the Hermite polynomials $H_N(x)$.[1] In a single formula, the solutions to (1) for both problems are simply given by

$$H_N(x_i) = 0 \tag{3}$$

The Hermite polynomial $H_N(x)$ is a polynomial of degree N solving the linear differential equation $P_N'' - 2xP_N' + 2NP_N = 0$. This fixes the polynomials up to an overall normalization which is irrelevant for this problem as it does not affect the location of its roots.

One and Two particles

1. [**2pt**] Verify the key theorem (3) for $N = 1$ and for $N = 2$ for both problems A and B.

Checking the key theorem for $N > 2$ is quite a bit harder! This is what we turn to next. We will establish it first for problem A. Then we will show that problem B admits the same solutions as problem A.

Problem A

We define $Q(z) \equiv \prod_{j=1}^{N}(z - x_j)$.

2. [**4pt**] Consider the ratio

$$r(z) = \frac{Q''(z)}{Q'(z)}\,. \tag{4}$$

Show that at $z = x_i$ this ratio simply evaluates to $r(x_i) = \sum_{j \neq i} g(x_i, x_j)$ and find g. This holds for arbitrary x_i. Show that if the x_i solve (1) for problem A then

$$r(x_i) = 2x_i \tag{5}$$

[1] You probably saw them when solving the Harmonic oscillator in Quantum Mechanics. The wave function of the n-th excited state is given by a gaussian times the Hermite polynomial of degree n.

3. **[4pt]** Consider the combination

$$P(z) = Q''(z) - 2zQ'(z) \tag{6}$$

with Q defined as above with x_i solving (1) for the problem A choice of $f(x) = 1/x$. Note that since Q is a polynomial, the combination $P(x)$ is also a polynomial. What is the degree of this polynomial and what are its roots?

4. **[4pt]** Show that $Q(z)$ obeys the differential equation

$$Q''(z) - 2zQ'(z) - \alpha Q(z) = 0\,. \tag{7}$$

Fix α and establish the main theorem (3) for problem A.
Hint: If you have two polynomials which are proportional to each other, you can easily fix the constant of proportionality by expanding them around a convenient point.

Problem B

5. **[2pt]** Both problems A and B can be derived as the extremization condition $\partial_{x_i}\mathcal{L}_{A,B} = 0$ for a Lagrangian

$$\mathcal{L}_X = \sum_i V(x_i) + \sum_{i<j} v_X(x_i - x_j)\,, \qquad X = A, B\,, \tag{8}$$

where $V(x) = x^2/2$. What is $v_A(x)$ for problem A? What is $v_B(x)$ for problem B?

6. **[5pt]** Show that for any set of roots x_i we have

$$\mathcal{L}_B = \frac{1}{2}\sum_i (\partial_{x_i}\mathcal{L}_A)^2 + N(N-1)/2 \tag{9}$$

Hint: What do we need to do to establish that two rational functions are the same?

7. **[4pt]** Prove theorem (3) for problem B.

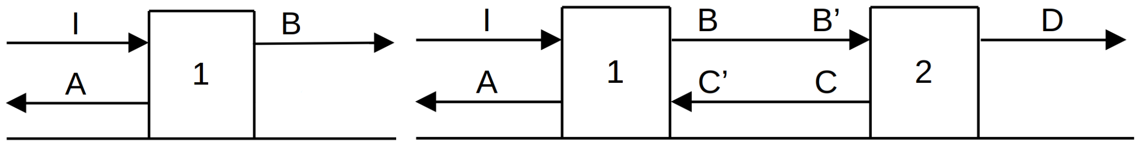

Figure 1: Left: the configuration with 1 scatterer. Right: the configuration with 2 scatterers.

2 Localization in 1D via the Landauer approach

In this problem, we are going to see a model for the phenomenon of *localization* in a $1D$ system, and we will reach this result via the so-called Landauer approach.

Let us first recall the S-matrix formulation for the problem of the scattering of a beam against an obstacle (see Fig. 1, left panel). A beam coming from the left, having amplitude I, gets partially reflected by the obstacle, giving rise to the left moving beam A, and is partially transmitted to the right of the obstacle with amplitude B. I, A, and B are connected by the S-matrix, which incorporates all the dynamics information of the scattering process. In detail, we have the following relations

$$\Psi_{\text{out}} = S \cdot \Psi_{\text{in}}, \qquad S \equiv \begin{pmatrix} r & t \\ t & r' \end{pmatrix}, \tag{10}$$

where the vector $\Psi_{\text{in}} \equiv (I,\, 0)^T$ denotes the incoming states (notice that we are assuming here that there is no incoming beam from the right of the barrier), the vector $\Psi_{\text{out}} \equiv (A,\, B)^T$ denotes the outgoing beams and the scattering coefficients r, t, and r', satisfy the following relations due to the requests of unitarity and time-reversal symmetry

$$\begin{aligned} & R \equiv |r|^2, \qquad T \equiv |t|^2, \qquad R' \equiv |r'|^2, \\ & R + T = 1 \\ & R = R', \\ & \frac{r}{(r')^*} = -\frac{t}{t^*}. \end{aligned} \tag{11}$$

In the Landauer approach, the *dimensionless* electrical resistance of the barrier is given by the formula

$$r_{\text{el}} = \frac{R}{T}. \tag{12}$$

Now consider the situation depicted in Fig. 1 on the right. An incoming wave, having amplitude denoted by I (which we will assume is fixed to $I = 1$), encounters a series of 2 scatterers, denoted by 1 and 2, respectively (and characterized by parameters r_i, t_i, and r_i', with $i = 1, 2$), and leaves them with amplitude D. Between the two scatterers, the beam propagates freely, and therefore we can assume

$$B' = Be^{i\phi} \quad C' = Ce^{i\phi}, \tag{13}$$

with ϕ being a coefficient related to the distance between the two scatterers and that we will get rid of at a later stage.

1. **[2pt]** Write a system of equations for A, B, C, and D, and use them to obtain the following expression for the outcoming beam (recall that we assume $I = 1$)

$$D = \frac{t_1 t_2 e^{i\phi}}{1 - e^{2i\phi} r_1' r_2}. \tag{14}$$

2. **[2pt]** Given that the total transmission coefficient T_{12} is $T_{12} = |D|^2$, write an expression for the electrical resistance, r_{el}, of the system of two scatterers, in terms of the reflection coefficients R_1 and R_2, the transmission coefficients T_1 and T_2, and the angle $\theta \equiv 2\phi \arg(r_2 r_1')$.

3. **[5pt]** Assuming that ϕ is a random uniformly distributed variable in the interval $[0, 2\pi]$, obtain a formula for the *averaged* resistance $\bar{r}_{\text{el}} \equiv \frac{1}{2\pi}\int_0^{2\pi} r_{\text{el}}(\theta)\mathrm{d}\,\theta$.

4. **[7pt]** Ohm's law says that the total resistance for passing by both barriers is the sum of the resistances of the two barriers. Does the total resistance $\bar{r}_{\text{el}}$ obtained at the previous point satisfy the Ohm law? In the negative case, is $\bar{r}_{\text{el}}$ larger or smaller than the Ohmic prediction?

5. **[4pt]** Now assume that we have a set of n identical scatterers, that are collectively characterized by a total averaged resistance $\bar{r}_{\text{el}}^{(n)}$. Using the results of the previous points, show that by adding an extra identical scatterer the resistance becomes

$$\bar{r}_{\text{el}}^{(n+1)} = \bar{r}_{\text{el}}^{(n)} + \bar{r}_{\text{el}}^{(1)} + 2\bar{r}_{\text{el}}^{(n)}\bar{r}_{\text{el}}^{(1)} \,. \tag{15}$$

6. **[5pt]** Use the above result to get a differential equation for the resistance of n scatterers, denoted by $\bar{r}_{\text{el}}(n)$, and show that the solution goes as

$$\bar{r}_{\text{el}}(n) \to e^{2\bar{r}_{\text{el}}^{(1)} n} \,, \tag{16}$$

when n is very large, *i.e.* that the resistance grows *exponentially* with the number of scatterers thus showing a very non-Ohmic behavior.

3 Length measures in different frames

In special relativity, length and time measures are observer dependent.

Consider a spacecraft moving in a 2-dimensional space $t - x$, its "rear" edge following the trajectory $x^2 - c^2t^2 = a_r^{-2}$, and its "front" edge following the trajectory $x^2 - c^2t^2 = a_f^{-2}$, with constant a_f and a_r.

You may find the integral $\int \frac{dt}{\sqrt{1+t^2}} = \text{ArcSinh(t)} = \log\left(t + \sqrt{t^2+1}\right)$ useful below. It behaves as $\log(2t)$ at large t.

1. [**1pt**] Draw the space-time trajectory of the spacecraft in coordinates t (vertical axis) and x (horizontal axis).

2. [**1pt**] Are the trajectories of the rear and front edges of the spacecraft time-like, light-like or space-like?

3. [**1pt**] Show that at $t = 0$ the spacecraft is spatially at rest with respect to the t, x pair of coordinates.

4. [**3pt**] What is the length of the spacecraft at rest in the t, x coordinates? Call it ΔL. What is the length of the spacecraft measured at fixed, generic t? Is it t dependent? How does it behave in the $t \to 0$ and $t \to +\infty$ limits?

5. [**5pt**] An astronaut travelling in the spacecraft measures the length of the spacecraft by using the radar technique. (S)he measures the time it takes for a light signal to travel from the rear of the spacecraft to a mirror in its front and finally collecting it at the rear. Explain why this length is time-independent.

   Hint: Use boost symmetry to show that any two radar measurements are related.

6. [**5pt**] By picking a suitable emission time, compute the length measured by an observer at the rear of the spacecraft in terms of a_f and a_r.

7. [**2pt**] Verify that the 2-velocities $(dt/d\tau, dx/d\tau)$ corresponding to the spacecraft rear and front points have the space-component proportional to t. Interpret the constant of proportionality as acceleration.

8. [**7pt**] Let us suppose the spacecraft will keep this trajectory until getting to Proxima Centauri, which is 4.2 light years from us. How much "terrestrial" time t would it take? How much time for the astronauts? Assuming $a_r c^2 \sim a_f c^2 = g \equiv 10m/sec^2$, would the astronauts be able to survive the travel without the need for hybernation?

   Hint: You might want to show – and use – the estimate $g \simeq c/(1\,year)$.

4 Aharanov-Bohm Effect

Although electromagnetism can be described classically in terms of the electric and magnetic field, $\vec{E}(t,\vec{x})$ and $\vec{B}(t,\vec{x})$, quantum electromagnetism also depends on the potential field $\vec{A}(t,\vec{x})$. As discussed below, this leads to the Aharanov-Bohm effect involving flux quantization.

1. **[1pt]** Suppose $\vec{B}$ satisfies $\vec{\nabla}\cdot\vec{B}=0$. Show that locally $\vec{B}=\vec{\nabla}\times\vec{A}$ for some $\vec{A}(t,\vec{x})$.
2. **[1pt]** Suppose $\vec{\nabla}\times\vec{E}=-\frac{d}{dt}\vec{B}$. Show that locally, $\vec{E}=-\frac{d}{dt}\vec{A}+\nabla\phi$ for some $\phi(t,\vec{x})$.
3. **[1pt]** Show that the definition of $\vec{A}$ and ϕ are ambiguous up to the local gauge transformation $\vec{A}\to\vec{A}+\vec{\nabla}\Lambda, \quad \phi\to\phi+\frac{d}{dt}\Lambda$ for any $\Lambda(t,\vec{x})$.
4. **[4pt]** Show that the non-relativistic equations of motion for a particle of mass m and charge q moving in an electromagnetic background,
$$m\frac{d^2}{dt^2}\vec{x}=q\Big(\vec{E}-\vec{B}\times\frac{d}{dt}\vec{x}\Big),$$
can be derived from the Lagrangian $L=\frac{1}{2m}\vec{P}\cdot\vec{P}-V(x)$ where $\vec{P}=m\frac{d}{dt}\vec{x}+q\vec{A}$. What is $V(x)$ as a function of ϕ and $\vec{A}$?

If $\Psi(t,\vec{x})$ is the wave function of a particle of charge q, the momentum operator is defined by $\vec{P}=-i\hbar\vec{\nabla}+q\vec{A}$.

5. **[4pt]** For the expectation value $\langle\vec{P}\rangle$ to be invariant, the wave function $\Psi(t,\vec{x})$ needs to also transform when one performs the gauge transformation of $\vec{A}$. What is the gauge transformation of $\Psi(t,\vec{x})$ which leaves $\langle\vec{P}\rangle$ invariant under $\vec{A}\to\vec{A}+\vec{\nabla}\Lambda$?
6. **[3pt]** Show that $[P_j,P_k]=-i\hbar q\epsilon_{jkl}B_l$. Explain how the wave function of the particle is affected if the particle takes different paths through a magnetic field.

Suppose one has a cylindrical solenoid of radius R and infinite length along the z axis and there is a constant B field pointing in the z direction inside the cylinder, i.e.

$$B_z(t,x,y,z)=f \quad \text{for} \quad x^2+y^2<R^2, \quad B_z(t,x,y,z)=0 \quad \text{for} \quad x^2+y^2\geq R^2.$$

7. **[2pt]** Show that $\oint_L d\vec{x}\cdot\vec{A}$ is non-zero for any closed path L around the cylinder.
8. **[2pt]** Show that $\vec{A}$ can be locally gauge-fixed to zero outside the cylinder, but the gauge parameter Λ cannot be globally defined to be single-valued.
9. **[5pt]** Show that $\Psi(t,\vec{x})$ can only be single-valued if the value of f takes certain values. What are the values of f such that $\Psi(t,\vec{x})$ is single-valued?
10. **[2pt]** A Dirac monopole at the point $\vec{x}_0$ can be interpreted as a solenoid of infinitesimal radius which stretches to infinity in one direction and ends at the point $\vec{x}_0$, i.e.
$$\vec{B}(t,\vec{x})=4\pi g\int^{\vec{x}_0} d\vec{x}'\delta^3(\vec{x}-\vec{x}')$$
where g is the magnetic charge of the monopole and the integral streches from ∞ to the point $\vec{x}_0$. What is the condition on g such that $\Psi(t,\vec{x})$ is single-valued?