IFT-Perimeter-SAIFR
Journeys into Theoretical Physics 2023
Sunday Exam

- Write your name on each page

- Number each page used to solve a given problem as $1/n, 2/n, \ldots, n/n$ where n is the number of pages used to solve that problem

- Do not solve more than one problem per page – these exams will be split apart and graded by different people.

- Problem 1 (Supersymmetry): 25%
- Problem 2 (The ground state of the hydrogen atom): 25%
- Problem 3 (Quantum Hall and the Monopole): 25%
- Problem 4 (Adding an extra doublet to the SM): 25%

◦ Full Name: ______________________________
◦ I am interested in applying to the IFT masters program even if I am not accepted into the PSI program: [Yes] [No]
◦ If accepted into any of the programs, I would be interested in starting my fellowship at the IFT in August 2023: [Yes] [No]
◦ The areas of physics which I am most interested in are:

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Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 Supersymmetry

Superspace in one dimension

Consider the "superfield" $X^m(\tau,\kappa) = x^m(\tau) + i\kappa\psi^m(\tau)$ which depends on one commuting variable τ and one anticommuting variable κ. Define $D = \frac{\partial}{\partial\kappa} + i\kappa\frac{\partial}{\partial\tau}$ to be a fermionic derivative.

1. **[2pt]** Compute the superfield $DX^m(\tau,\kappa)$ in terms of $x^m(\tau)$ and $\psi^m(\tau)$
2. **[2pt]** Show that $D^2 = i\frac{\partial}{\partial\tau}$
3. **[6pt]** Perform the integration over κ to write the Lagrangian L in terms of $x^m(\tau)$, $\psi^m(\tau)$ and $A_m(x)$ in the expression

$$L = -i\int d\kappa\ [\frac{M}{2}\frac{\partial X_m}{\partial\tau}DX^m + \frac{q}{c}A_m(X)DX^m]$$

Irreducible representations of the Lorentz group

Consider the fields $f_{(ab)}(x)$ and its complex conjugate $\bar{f}_{(\dot{a}\dot{b})}(x)$ which are symmetric in the spinor indices.

4. **[1pt]** How many independent components are described by $f_{(ab)}$ and $\bar{f}_{(\dot{a}\dot{b})}$?
5. **[3pt]** Using the help of Pauli matrices $\sigma^m_{a\dot{a}}$ and the antisymmetric tensors $\epsilon_{[ab]}$ and $\epsilon_{[\dot{a}\dot{b}]}$, show that any antisymmetric tensor with two vector indices, $f_{[mn]}(x)$, can be expressed in terms of $f_{(ab)}(x)$ and $f_{(\dot{a}\dot{b})}(x)$.

Consider the field $g_{(ab)(\dot{a}\dot{b})}(x)$ which is symmetric in the chiral indices (ab) and also symmetric in the antichiral indices $(\dot{a}\dot{b})$.

6. **[1pt]**How many independent components are described by $g_{(ab)(\dot{a}\dot{b})}$?
7. **[5pt]** Using the help of Pauli matrices $\sigma^m_{a\dot{a}}$, express $g_{(ab)(\dot{a}\dot{b})}(x)$ in terms of a tensor with only vector indices and describe the properties of this tensor so that it has the correct number of independent components.

Spinors

8. **[5pt]** The equations of motion for a Majorana spinor are

$$\sigma^m_{a\dot{a}}\partial_m\bar{\psi}^{\dot{a}} = iM\psi_a, \quad (\bar{\sigma}^m)^{\dot{a}a}\partial_m\psi_a = iM\bar{\psi}^{\dot{a}}.$$

Using the identity $\sigma^m_{a\dot{a}}(\bar{\sigma}^n)^{\dot{a}b} + \sigma^n_{a\dot{a}}(\bar{\sigma}^m)^{\dot{a}b} = 2\eta^{mn}\delta^b_a$, show that these equations imply

$$(\partial^m\partial_m + M^2)\psi_a = 0.$$

2 The ground state of the hydrogen atom

The Hamiltonian of the hydrogen atom is given by

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e_e^2}{r} \tag{1}$$

where r is the radial distance from the nucleus to the position of the electron, e_e, m are the electron's charge and mass. You will use the trial wavefunction $\Psi_\alpha(\boldsymbol{r}) = Ce^{-\alpha r}$, where C is the normalization constant, and α is a variational parameter, to estimate the ground state energy of the hydrogen atom.

1. [**5 pts**]. Evaluate the normalization constant C by enforcing that $\int d\boldsymbol{r}|\Psi_\alpha(\boldsymbol{r})|^2 = 1$

2. [**10 pts**] Using the normalized wavefunction, prove that the expectation value of the energy

$$E(\alpha) = \langle\Psi_\alpha|H|\Psi_\alpha\rangle = \int d\boldsymbol{r}\, \Psi_\alpha(\boldsymbol{r})^*\, H\Psi_\alpha(\boldsymbol{r}) = \frac{\hbar^2\alpha^2}{2m} - \alpha e_e^2 \tag{2}$$

3. [**2 pts**] Find the value α^* that minimizes the variational energy $E(\alpha)$ by direct solution. Evaluate $E(\alpha^*)$ and compare it with the exact value of the ground state energy of the hydrogen atom $E_{\text{exact}} = -\frac{m\,e_e^4}{2\,\hbar^2}$.

4. [**3 pts**] Now, instead of a direct solution to the minimization problem, you will use gradient descent techniques to solve the minimization of the energy problem. Instead of traditional gradient descent, you will use the gradient flow, which is the continuous-time version of gradient descent, where we take the limit of learning rate going to zero. The gradient flow update is

$$\frac{d\alpha}{dt} = -\frac{dE(\alpha)}{d\alpha} \tag{3}$$

   Show that under the gradient flow, the energy is a decreasing function of time, i.e., $\frac{dE(\alpha)}{dt} \leq 0$.

5. [**5 pts**] Show that as a function of time, the gradient flow of α converges exponentially fast to the optimum α^*.

Useful expressions:

1. $\int_0^\infty dx\, x^n e^{-x\,a} = \frac{\Gamma(n+1)}{a^{n+1}}$

2. $\Gamma(n) = (n-1)!$ for positive integers n.

3. The laplacian in spherical coordinates is given by

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \quad (4)$$

   where f is the function for which the Laplacian is being computed. r, θ, and ϕ are the radial, polar, and azimuthal coordinates, respectively.

4. The solution to a differential equation of the form: $\frac{df(x)}{dx} = -a(f(x) - b)$ is given by $f(x) = Ke^{-ax} + b$ where K is a constant.

5. A three dimensional integral in spherical coordinates looks like this:

$$\iiint f(r, \theta, \phi)\, r^2 \sin\theta \, dr \, d\theta \, d\phi \quad (5)$$

3 Quantum Hall and the Monopole

1. [**5pt**] Write down the Hall resistance of the integer quantum Hall effect and briefly describe its physical mechanism. In particular, you should explain 1) why there are plateaus; 2) why the Hall resistance at each plateau is quantized. (You do not need to derive any formula.)

2. Landau levels on a sphere (we set $\hbar = c = e = 1$ in this probelm). We consider electrons moving on a sphere with the radius R, and we place a magnetic monopole with $4\pi s$ flux in the center of sphere. The magnetic field created by the monopole is $\vec{B} = \frac{s}{R^2}\vec{e}_r$, and the single particle Hamiltonian of the electron is

$$H = \frac{\vec{\Lambda}^2}{2MR^2}, \tag{6}$$

where $\vec{\Lambda} = \vec{r} \times (-i\vec{\nabla} + \vec{A}(\vec{r}))$ is the mechanical angular momentum, and $\vec{r} = R\vec{e}_r$.

   (a) [**5pt**] Prove the Dirac quantization condition $2s \in \mathbb{Z}$. (Hint: Use a similar argument that demonstrates the Chern number is an integer.)

   (b) [**5pt**] We define an operator, $\vec{L} = \vec{\Lambda} + s\,\vec{e}_r$. Show that it follows the $SO(3)$ angular momentum algebra,

$$[L^i, L^j] = i\epsilon^{ijk}L^k, \tag{7}$$

   where $i, j, k = x, y,$ or z, and ϵ^{ijk} is an antisymmetric tensor, i.e., $\epsilon^{ijk} = -\epsilon^{jik}$, $\epsilon^{ijk} = \epsilon^{jki}$, and $\epsilon^{xyz} = 1$.

3. [**10pt**] Compute the energies of each Landau level. (Hint: Express the Hamiltonian in terms of $\vec{L}$.)

Useful formulas

$\vec{e}_r = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$, $\vec{e}_\theta = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, -\sin\theta)$, $\vec{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$.
$\vec{e}_r \times \vec{e}_\theta = \vec{e}_\varphi$, $\vec{e}_\theta \times \vec{e}_\varphi = \vec{e}_r$, $\vec{e}_\varphi \times \vec{e}_r = \vec{e}_\theta$. The nabla operator is,

$$\vec{\nabla} = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}. \tag{8}$$

4 Adding an extra doublet to the SM

One of the simplest extensions of the Standard Model one can consider is adding an extra scalar doublet (under SU(2)), obtaining the so called two-Higgs Doublet model (2HDM). In these models there are two scalars, ϕ_1 and ϕ_2, both tranforming under $SU(2)_L \times U(1)_Y$ as:

$$\phi_{1,2} \to e^{\frac{i\alpha_a \sigma_a}{2}} e^{\frac{i\alpha}{2}} \phi_{1,2} \tag{9}$$

where $a \in \{1, 2, 3\}$, $\alpha_a = \alpha_a(x)$ and $\alpha = \alpha(x)$ are the parameters of a **local** transformation (and thus, functions of x_μ) and σ_a are the Pauli matrices.

1. **[2pt]** Write down convariant derivatives for these two fields.

2. **[5pt]** Write the most general potential for these two fields that respects the $SU(2)_L \times U(1)_Y$ symmetry. By potential we mean the part of the Lagrangian that only involves ϕ_1 and ϕ_2 and contain no derivatives. Limit yourself to terms involving up to four powers of these fields (the so called "relevant" operators).

3. **[8pt]** Assuming that the potential you just wrote accepts, for some combination of its parameters, a minimum such that the vacuum expectation value of the two scalar fields is:

$$\langle \phi_{1,2} \rangle_0 = \begin{pmatrix} 0 \\ v_a/\sqrt{2} \end{pmatrix} \tag{10}$$

   choose a parametrization for the 8 degrees of freedom contained in these two fields and re-write the Lagrangian so that it is clear that we have 3 real massive scalars and 1 complex massive scalar (which means we have 3 massless Goldstone Bosons, which will give mass to the W's and the Z).

4. **[8pt]** Starting from the kinetic terms $\mathcal{L}_{\text{kin}} = |D_\mu \phi_1|^2 + |D_\mu \phi_2|^2$, calculate the masses of the W and Z bosons in this model.

5. **[2pt]** Considering the masses calculated in the previous question, what condition should be satisfied by the VEVs in the 2HDM so that it is consistent with the well known masses for the W and Z?