IFT-Perimeter-SAIFR

Journeys into Theoretical Physics 2023

Saturday Exam

- Write your name on each page

- Number each page used to solve a given problem as $1/n, 2/n, \ldots, n/n$ where n is the number of pages used to solve that problem

- Do not solve more than one problem per page – these exams will be split apart and graded by different people.

- Problem 1 (Relativistic Particle in Constant Electromagnetic Field): 25%
- Problem 2 (Renormalization in Quantum Mechanics): 25%
- Problem 3 (Car Jams and Phase Transitions): 25%
- Problem 4 (Confining Gravitational Motion): 25%

○ Full Name: ______________________________

○ I am interested in applying to the IFT masters program even if I am not accepted into the PSI program: [Yes] [No]

○ If accepted into any of the programs, I would be interested in starting my fellowship at the IFT in August 2023: [Yes] [No]

○ The areas of physics which I am most interested in are:

____

Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 Relativistic Particle in Constant Electromagnetic Field

Consider a particle of mass m and charge q with velocity $\vec{v}$ moving in a constant electromagnetic background described by electric and magnetic fields $\vec{E}$ and $\vec{B}$ which are independent of space and time. The equation of motion for the particle is determined by the force law

$$\vec{f} = q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B}\,. \tag{1}$$

1. **[2pt]** Suppose that $\vec{E} = (0, 0, E)$ and $\vec{B} = (0, 0, 0)$. If the particle at time $t = 0$ is at the position $x = y = z = 0$ with velocity $\vec{v} = (0, v_0, 0)$, what is the trajectory $(x(t), y(t), z(t))$ for a non-relativistic particle when $|v| << c$?

2. **[2pt]** Now suppose that $\vec{E} = (0, 0, 0)$ and $\vec{B} = (B, 0, 0)$. If the particle at time $t = 0$ is at the position $x = y = z = 0$ with velocity $\vec{v} = (0, v_0, 0)$, what is the trajectory $(x(t), y(t), z(t))$ for a non-relativistic particle when $|v| << c$?

The relativistic equation of motion for the particle is given by

$$\frac{d}{dt}P_\mu = \frac{q}{c}\frac{dX^\nu}{dt}F_{\mu\nu}\,, \tag{2}$$

where $P_\mu = m\gamma\frac{dX_\mu}{dt}$ is the relativistic momentum, $\gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$, $X^\mu = (ct, x, y, z)$ for $\mu = 0$ to 3, and $F_{\mu\nu}$ is the constant electromagnetic background defined by $F_{j0} = -F_{0j} = E_j$ and $F_{jk} = \epsilon_{jkl}B_l$ for $j = 1$ to 3. The signature convention used is $(-, +, +, +)$.

3. **[2pt]** Show that the force law of eq (1) can be obtained from the relativistic eq (2).

4. **[8pt]** Redo question 1 for the relativistic case. <u>Hint:</u> First use eq (2) to find $\gamma(t)$, and then find dX^μ/dt. To obtain $X^\mu(t)$, the formula $\frac{d(\sinh^{-1}(At))}{dt} = A(1 + (At)^2)^{-\frac{1}{2}}$ may be useful.

5. **[2pt]** Show that the relativistic case of question 4 reduces to the non-relativistic case of question 1 when $|v| << c$.

6. **[5pt]** Redo question 2 for the relativistic case. <u>Hint:</u> First show that γ is independent of t.

7. **[2pt]** Consider $\vec{E}$ and $\vec{B}$ to be constant perpendicular vectors. Under what conditions can one boost to another referential frame where the electric field vanishes? What is the magnitude of $\vec{B}$ in that case? <u>Hint:</u> $F_{\mu\nu}F^{\mu\nu}$ is a Lorentiz invariant.

8. **[2pt]** Show that if E and B are constant vectors which are not orthogonal we can not go to a frame without electric or magnetic field. <u>Hint:</u> Consider another simple Lorenz invariant in four dimensions involving the fully antisymmetric Levi-Civita symbol $\epsilon_{\mu\nu\rho\delta}$.

2 Renormalization in Quantum Mechanics

In this exercise we will explore the Hamiltonian

$$H = \sum_{n=0}^{\infty} E_n |n\rangle\langle n| - g|e\rangle\langle e| \tag{3}$$

where $E_n = \omega n$, g is a small positive real number and $|e\rangle$ is a normalized state in the Hilbert space which can be decomposed into a *finite* number of $|n\rangle$'s,

$$|e\rangle = \sum_{n=0}^{N} e_n |n\rangle \tag{4}$$

where e_n are **real** numbers. The discussion below will not depend on N but it is important that N is finite; you can take it to be 4 say. In this problem we will consider the eigenvalue problem

$$H|\psi\rangle = E|\psi\rangle \tag{5}$$

for the eigenstates $|\psi\rangle = \sum_k \psi_k |k\rangle$.

Suggestion: Read the full problem before starting to answer as the answer to some questions can be hinted at by the next questions...

The Eigenvalue Problem

1. [**0.5pt**] Show that the eigenvalue equation in terms of ψ_k is

   $$(E_k - E)\psi_k = \gamma_E e_k \tag{6}$$

   where $\gamma_E = g\langle e|\psi\rangle$.

   Now we have two situations

2. [**3.5pt**] Case one: For some l, $e_l = 0$. What can you conclude about the eigenvalue and eigenstate corresponding to this l? How is it related to the original eigenvalue and eigenstate? Can you also show the converse of this statement is also true?

   Hint: Start the reasoning with, if the eigenvalue is such and such, suppose $e_l \neq 0$ and try to reach a contradiction.

   From this we learn that the eigenvalue structure of H agrees with that of H_0 whenever $e_l = 0$.

3. [**0.5pt**] Case two: Now let us look at the other case when $E_k \neq E$. In this case, we can immediately solve for ψ_k. What is ψ_k?

4. [**1pt**] Now let us use the normalization condition of $|\psi\rangle$, which allows us to find γ_E. What is γ_E?

5. **[2pt]** Use γ_E's definition derive the characteristic equation

$$g\sum_{k=0}^{N}\frac{e_k^2}{E_k - E} = 1. \tag{7}$$

6. **[3pt]** Now let us suppose $e_0 \neq 0$ [1]. Consider $E < 0$, how does the left hand side of the characteristic equation change as we increase E? What is the range of value for the left hand side of the characteristic equation? What can you say about the solution of the characteristic equation when $E < 0$?

7. **[2.5pt]** Now for $e_k \neq 0$, consider $E_k < E < E_{k+1}$[2]. Can you make a similar argument about the left hand side of the characteristic equation as before? What can we say about the solution?

   These are the ones we found in case two.

8. **[3pt]** So what is the effect of the interaction? What happens to the eigenvalues when $e_k = 0$? What happens to the eigenvalues when $e_k \neq 0$? Do they increase, decrease, or stay the same? Does it make sense?

   Hint: compare to the form of the interacting Hamiltonian.

Renormalization

There is a problem when we consider our characteristic equation at tiny coupling constant g and large $\sum_{E_k \neq 0} \frac{e_k^2}{E_k}$. Focus on $E < 0$ and approach the first eigenvalue – the negative one we found above – from the left.

9. **[3pt]** Show that a small error in g could now lead to a big error in the energy determination of this first eigenvalue.

   Thus we should change our variable to something the solution does not so sensitively depend on. Here is a proposal:

$$g = g(\Delta) \equiv \left(\sum_{k=0}^{N}\frac{e_k^2}{E_k + \Delta}\right)^{-1} \tag{8}$$

   where $\Delta > 0$.

10. **[2pt]** What is the range of this function? And what other property does it have so that our problem can be described by Δ instead of g?

11. **[2pt]** Put this back in the characteristic equation. What is an obvious eigenvalue? How could we interpret Δ physically?

[1] the argument still works if $e_0 = 0$, we simply discard the first few 0 cases until we have a non-zero

[2] Assuming $e_{k+1} \neq 0$. If it is not again we can discard the zero case and take the first non-zero case.

12. **[2pt]** Show that the new renormalized g independent characteristic equation for nontrivial eigenvalue is

$$\sum_{k=0}^{N} \frac{e_k^2(E+\Delta)}{(E_k+\Delta)(E_k-E)} = 0. \tag{9}$$

Because it is g independent, the previous problem of sensitivity respect to g is solved – for all eigenvalues in fact!

We can make the same argument about the sliding eigenvalues as before and they are *not* very sensitive to Δ. It was thus a good idea to renormalize g and switch to a more physical measure of the coupling given by Δ.

3 Car Jams and Phase Transitions

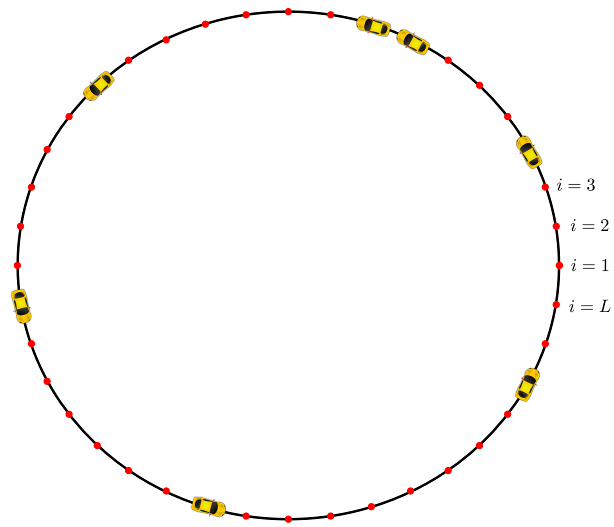

Figure 1: Cars start at locations $x_i(0) < x_{i+1}(0)$ on the circle and move without collisions.

Consider a sequence of cars on a one dimensional circular ring as depicted in figure 1. Time, the location x_i of car i and its velocity v_i will all be discrete in this problem:

$$t \in \{0, 1, 2, 3, \dots\}\,, \qquad x_i(t) \in \{1, 2, 3, \dots, L\}\,, \qquad v_i(t) \in \{0, 1, 2, 3, \dots, v_{max}\}\,. \tag{10}$$

At time $t = 0$ we start with a random distribution of N cars on the circle with $x_i(0) < x_{i+1}(0)$ each with a random velocity $v_i(0) \in \{0, 1, 2, 3, \dots, v_{max}\}$. Then as t moves in integer steps, the car locations and velocities are updated according to

$$x_i(t+1) = [x_i(t) + v_i(t+1)] \tag{11}$$

where $[\dots]$ stands for $\dots \bmod L$ (in other words, it is the distance on the ring) and where

$$v_i(t+1) = \max\left(0, -r_i(t) + \begin{cases} \min(v_i(t)+1, v_{\max}) & \text{if } [x_{i+1}(t) - x_i(t)] > v_i(t) + 1 \\ [x_{i+1}(t) - x_i(t)] - 1 & \text{if } [x_{i+1}(t) - x_i(t)] \leq v_i(t) \\ v_i(t) & \text{otherwise} \end{cases}\right)$$

where $r_i(t) = 1$ with probability p and $r_i(t) = 0$ with probability $1 - p$.

1. **[4pt]** How do you interpret the position and velocity updating rules?
2. **[2pt]** Explain why cars never crash in this model.

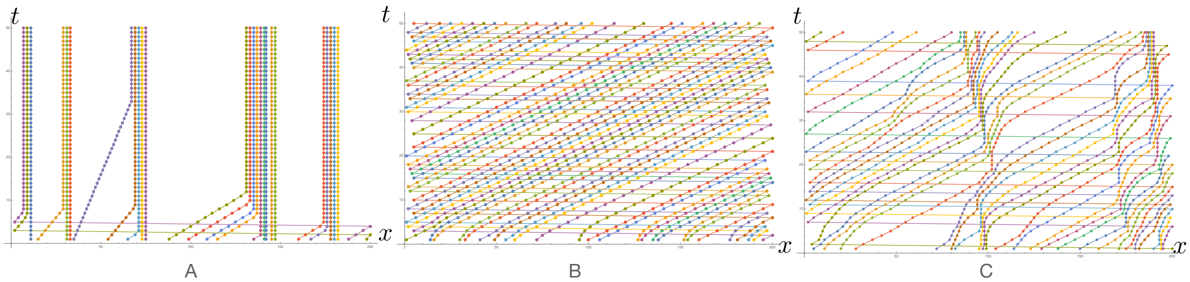

Figure 2: Three simulations where cars move on a ring. Time moves up.

3. **[2pt]** In figure 2 we depict the outcomes of three different simulations simulations with $L = 200$, $N = 25$, $v_{\max} = 5$ and different breaking probability $p = 0, 1/2$ and 1. Which simulation in the figure corresponds to each of these p values? Explain.

4. **[3pt]** Let $n_i(t) = 1$ if site i is occupied at time t and 0 otherwise. Let $m_i(t) = 1$ if a car passed through site i at time t, and 0 otherwise. More precisely $m_i(t) = 1$ is there was a car at $X \leq i$ at t which is at $Y > i$ at $t + 1$. Then we define

$$\rho_i(T) = \frac{1}{T}\sum_{t=0}^{T} n_i(t)\,, \qquad q_i(T) = \frac{1}{T}\sum_{t=0}^{T} m_i(t)\,. \tag{12}$$

What is the meaning of these quantities?

5. **[2pt]** In figure 3 we depict histogram of ρ_i and q_i for a given simulation for three different times $T = 50, 200, 4000$. Which curve is which? Explain your answer. What do we expect for the i dependence of these quantities as $T \to \infty$?

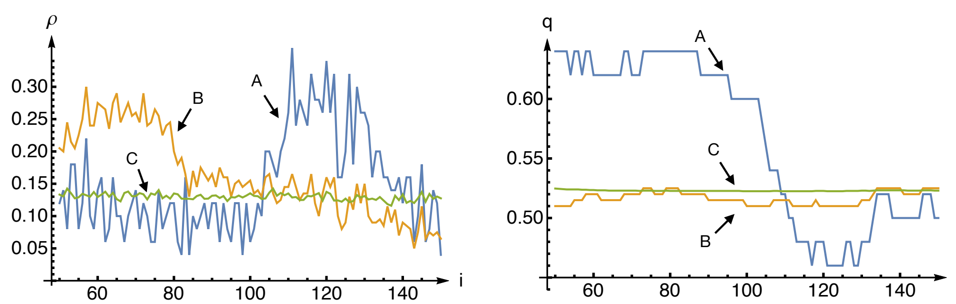

Figure 3: For a same simulation we plot ρ_i and q_i for $i = L/4, \ldots 3L/4$. The three curves correspond to three different T's.

6. **[2pt]** Estimate the number of cars N for the simulations depicted in figure 3.

7. **[4pt]** Let ρ and q be the asymptotic average of ρ_i and q_i as $T \to \infty$ and

$$M \equiv 1 - \frac{q}{v_{max}\rho} \tag{13}$$

be an order parameter. What range can M take? How would you describe a car flow with $M = 1$? What about $M = 0$?

Figure 4 depicts M for several car densities ρ and breaking probabilities p.

8. For $p = 0$ one can actually derive the exact curve $M(\rho)$. It is given by

$$M = \begin{cases} 0 & \text{for } \rho < \rho_c \\ A\frac{\rho-\rho_c}{\rho\rho_c} & \text{for } \rho \geq \rho_c \end{cases} \tag{14}$$

   (a) **[3pt]** What is the value of the constant A as a function of the critical density ρ_c?

   (b) **[3pt]** What is the value of the critical density ρ_c as a function of the maximum velocity $v_{\max}$?

   Hint: Think of the last most packed free stationary flow before a jam forms.

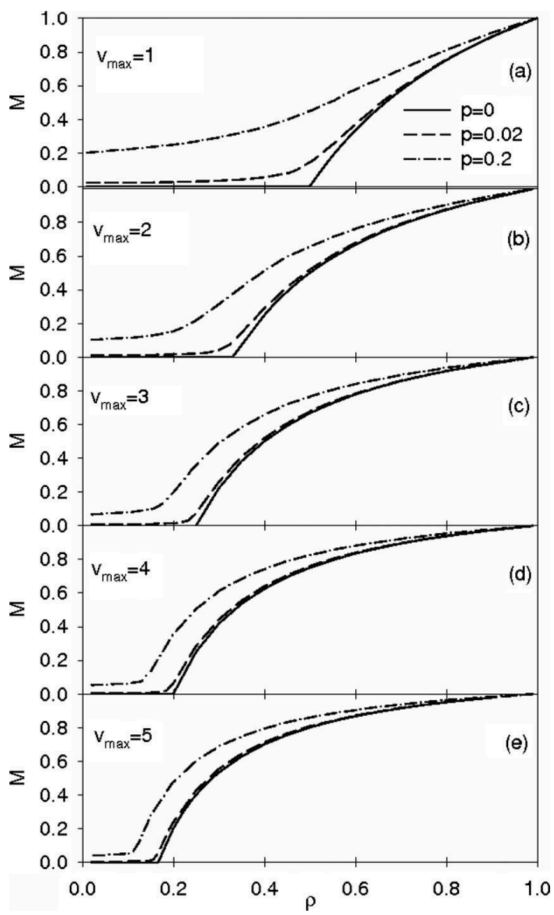

Figure 4: $M(\rho)$ for various simulations with L and N very large (*"thermodynamic"* limit).

4 Confining Gravitational Motion

Consider a system of N non-relativistic particles that interact via an *attractive* force proportional to the distance between particles

$$\vec{F}_{ji} = -k m_i m_j (\vec{r}_i - \vec{r}_j) \tag{15}$$

where $i, j = 1, 2, \ldots, N$ and $m_i > 0$ is the mass of particle i. To be clear, $\vec{F}_{ji}$ is the force exerted on particle i by particle j. At the initial time $t = 0$ these particles have some given initial positions $\vec{r}_i^{(0)}$ and velocities $\vec{v}_i^{(0)}$.

1. **[5pt]** Write down the equation of motion for particle i. What is the sign of k?
2. **[5pt]** What is the acceleration of the center of mass $\vec{R}$? Derive that result from the equation of motion of the previous point. How will $\vec{R}$ evolve in time?
3. **[2pt]** Show that the total angular momentum $\vec{L} = \sum_i m_i(\vec{r}_i - \vec{r}_0) \times \dot{\vec{r}}_i$ – around any constant reference point $\vec{r}_0$ – is conserved. When is $\vec{L}$ independent of the reference point $\vec{r}_0$?
4. **[2pt]** Is the kinetic energy $K = \sum_i \frac{m_i}{2} \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$ conserved? If yes, why? If no, which quantity involving the kinetic energy is conserved?
5. **[3pt]** How do the equations of motion look like in the center of mass (CM) frame?
6. **[8pt]** Show that in the CM frame all particles follow ellipses with the same time period T. What is T? What is the center of the ellipses? What sets the shape of the ellipses?