Question 1.

Consider a change of coordinates

$$F = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3. \tag{1}$$

Show that

$$F^*(\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3) = \mathrm{Jac}(F)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3. \tag{2}$$

Convince yourself that the same statement continues to hold in n dimensions.

Question 2.

Give $\mathbb{R}^2$ coordinates x_1, x_2 and $\mathbb{R}^3$ coordinates y_1, y_2, y_3. Let $U \subset \mathbb{R}^2$ be the region where $y_1^2 + y_2^2 < 1$. Consider the mapping

$$F : (y_1, y_2) \mapsto (y_1, y_2, \sqrt{1 - y_1^2 - y_2^2}). \tag{3}$$

This map parameterizes the hemi-sphere

$$H = \{\sum x_i^2 = 1, \;\; x_3 > 0\} \tag{4}$$

with the coordinates y_1, y_2.

Let ω be the two-form

$$\omega = x_1 \mathrm{d}x_2 \mathrm{d}x_3 - x_2 \mathrm{d}x_1 \mathrm{d}x_3 + x_3 \mathrm{d}x_1 \mathrm{d}x_2 \tag{5}$$

Calculate the two-form $F^*\omega$ on the disc with coordinates y_1, y_2. Hence write the integral $\int_H \omega$ as an integral over the disc with coordinates y_1, y_2. (No need to calculate the integral).

Question 3.

Consider the open subset $U \subset \mathbb{R}^2$ defined by

$$U = \{(x_1, x_2) \mid (x_1, x_2) \neq (0,0) \;\; \text{and} \; (x_1, x_2) \neq (1,0)\}. \tag{6}$$

Use the Mayer-Vietoris sequence to show that $H^1(U)$ is two dimensional. Write down closed 1-forms which represent these cohomology classes.