How to evaluate such integrals?

* easy when $S[\varphi]$ is quadratic in φ, e.g.

$$\frac{1}{Z}\int \mathcal{D}\varphi \; \varphi(x_1)\,\varphi(x_2)\,\varphi(x_3)\,\varphi(x_4)\; e^{iS[\varphi] + i\int d^4x\, J(x)\varphi(x)} =$$

$$= \frac{1}{\cancel{Z}}\,\frac{1}{(i)^4}\,\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_4)}\;\underbrace{\int \mathcal{D}\varphi\; e^{iS[\varphi] + i\int d^4x\, J(x)\varphi(x)}}_{\cancel{Z}\; e^{iW[J]}}$$

$$= \frac{1}{(i)^4}\,\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_4)}\; e^{iW[J]} \qquad \text{------ (139)}$$

$$\sim D(x_1 - x_2)\,D(x_3 - x_4) + D(x_1 - x_3)\,D(x_2 - x_4)$$

$$+ D(x_1 - x_4)\,D(x_2 - x_3) \qquad \text{---------- (140)}$$

$\uparrow$ Wick's theorem

* $S[\varphi]$ is not quadratic $\rightarrow$ interactions

- when applicable, use perturbation theory

$\Downarrow$

$$S[\varphi] = S_0[\varphi] + S_{int}[\varphi] \qquad \text{----- (141)}$$

$\uparrow$ quadratic in φ $\qquad$ $\uparrow$ non quadratic

$$e^{i S_0[\varphi] + i S_{int}[\varphi]} = e^{i S_0[\varphi]} \Big[1 + i S_{int}[\varphi]$$

$$+ \frac{1}{2!} \left(S_{int}[\varphi] \right)^2 + \frac{1}{3!} \left(S_{int}[\varphi] \right)^3 + \cdots \Big] \quad \text{--- } (142)$$

Therefore, a N-point correlation function can be evaluated as follows:

$$C(x_1, \cdots, x_N) = \frac{1}{Z} \int \mathcal{D}\varphi \, \varphi(x_1) \cdots \varphi(x_N) \, e^{i S[\varphi]}$$

$$= \frac{1}{Z} \int \mathcal{D}\varphi \, \varphi(x_1) \cdots \varphi(x_N) \, e^{i S_0[\varphi]}$$

$$+ \frac{1}{Z} \int \mathcal{D}\varphi \, \varphi(x_1) \cdots \varphi(x_N) \left(i S_{int}[\varphi] \right)$$

$$+ \frac{1}{Z} \int \mathcal{D}\varphi \, \varphi(x_1) \cdots \varphi(x_N) \, \frac{1}{2!} \left(i S_{int}[\varphi] \right)^2$$

$+ \cdots$

Note: have to expand the denominator Z as well

$$= C^{(0)}(x_1, \cdots, x_N) + C^{(1)}(x_1, \cdots, x_N) + \cdots$$

$$\text{------- } (143)$$

$C^{(0)}(x_1, \cdots, x_N) \sim D(x_1 - x_2) D(x_3 - x_4) + \cdots$ derived in Eq. (140)

Suppose that the interaction term $S_{int}[\varphi]$ is

$$S_{int}[\varphi] \hat{=} \int d^4x\, \mathcal{L}_{int}(\varphi), \quad \mathcal{L}_{int} = -\tfrac{1}{4!}\lambda\left(\varphi(x)\right)^4$$

1st order term : $C^{(1)}(x_1, \dots, x_N)$

$$\int \mathcal{D}\varphi\, \varphi(x_1)\cdots\varphi(x_N)\left[-\tfrac{1}{4}\lambda\int d^4x\left(\varphi(x)\right)^4\right] e^{iS_0[\varphi]} =$$

$$= -\frac{1}{4}\lambda\int d^4x \int \mathcal{D}\varphi\, \underbrace{\varphi(x_1)\cdots\varphi(x_N)\left(\varphi(x)\right)^4}_{\varphi(x_1)\cdots\varphi(x_N)\,\varphi(x)\,\varphi(x)\,\varphi(x)\,\varphi(x)} e^{iS_0[\varphi]} \quad \text{---- (141)}$$

use Wick's theorem

– sum over all pairwise contractions

$\downarrow$ N even $\to$ ok

N odd $\to$ zero, in absence of SSB

Note, however: in the contractions, there are terms in which $\varphi(x_1)\cdots\varphi(x_N)$ are not contracted with the $\varphi(x)\,\varphi(x)\,\varphi(x)\,\varphi(x)$

$\Rightarrow$ they are "disconnected"

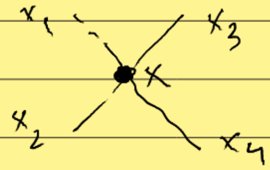
Example: $N = 4$

$$\varphi(x_1)\,\varphi(x_2)\,\varphi(x_3)\,\varphi(x_4)\,\varphi(x)\,\varphi(x)\,\varphi(x)\,\varphi(x)$$

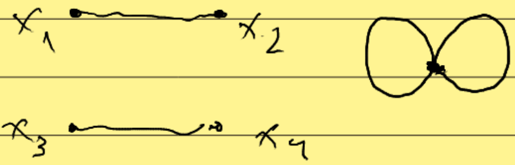
$$\rightarrow D(x_1 - x)\,D(x_2 - x)\,D(x_3 - x)\,D(x_4 - x)$$

$$+ \cdots + \left[D(x_1 - x_2)\,D(x_2 - x_4) + \cdots \right] D(x - x)\,D(x - x) \qquad (142)$$

1st line: connected



2nd line: disconnected



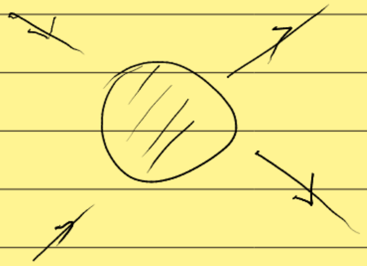
BUT: the disconnected cancell when expanding the $\frac{1}{Z}$ in powers of λ

– the cancellation happens order-by-order

$$\Downarrow$$

$$\frac{1}{Z} \int \mathcal{D}\varphi\, \varphi(x_1) \ldots \varphi(x_N)\, e^{iS} \Longleftarrow \text{only connected Terms contribute} \qquad (143)$$

Scattering amplitude: 2 particles $\longrightarrow$ 2 particles



$$A(p_1, p_2; p_3\, p_4) \to {}_{+\infty}\langle p_3\, p_4 | p_1\, p_2 \rangle_{-\infty}$$

Theorem: will not prove it, see e.g. Luke's lectures

$\hookrightarrow$ LSZ reduction formula:

$$i A(p_1, p_2; p_3\, p_4)\, (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - p_3 - p_4) = C(p_1, p_2, p_3, p_4)\Big|_{\text{on-shell}}^{\text{amp.}} \qquad (144)$$

where $C(p_1, \dots, p_4)$ is the Fourier transform

$$C(p_1, \dots, p_4) = \int d^4x_1 \dots d^4x_4\, e^{-i(p_1 \cdot x_1 + p_2 \cdot x_2 - p_3 \cdot x_3 - p_4 \cdot x_4)}\, C(x_1, \dots, x_4)$$

with $C(x_1, \dots, x_4)$ being the 4-point correlation function:

$$C(x_1, \dots, x_4) = \frac{1}{Z} \int \mathcal{D}\varphi\, \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)\, \varphi(x_4)\, e^{iS[\varphi]} \qquad (145)$$

amp. in Eq. (144) means amputation of external propagators (= depend only on $p_1, \dots, p_4$)

on-shell in Eq. (144) means put all external particles on-shell : $p_i^2 = m^2$

Exercice : show that for $\mathcal{L}_{int} = -\frac{1}{4!}\lambda\varphi^4(x)$:

$$\mathcal{A}(p_1, \dots, p_4) = -\lambda \qquad \text{-----} (146)$$

to first order in λ.

Stewart

Construction of an EFT

* Determine the relevant degrees of freedom

↳ not always easy, e.g. physics at the confinement scale in QCD

What fields?

** Identify the symmetries

↳ also not always easy, e.g. emergent symmetries due SSB (smallness of pion masses)

What interactions?

*** Find an expansion parameter and get an first order description

What power counting to use

Ready to write down the Lagrangian of your EFT

Construction of the Standard model:

- essentially followed * and **
- not much *** ← but extremely important in EFT

Key principle: to describe physics at scale m^2, do not need to know dynamics at scale $M^2 \gg m^2$

field content
symmetries

Generically: (scalar field only)

The Lagrangian is written as

$$\mathcal{L}_{EFT} = \sum_n C_n(\Lambda)\, O^{(n)}(\varphi, \partial\varphi) \quad ---- (147)$$

where $O^{(n)}$ is polynomial in φ and $\partial\varphi$

n: mass dimension of $O^{(n)}$, $[\text{mass}]^n$

Λ: cutoff, large mass scale

Mass dimension of a field: $\hbar = 1$, $c = 1$

$\hbar c \simeq 200$ MeV fm, $[\text{mass}] =$ MeV or fm^{-1}

Dimension of the action

$$e^{i/\hbar\, S} \rightarrow [S] = [\hbar] = [1] \rightarrow \underbrace{(\text{mass})^0}_{n=0} \quad --- (148)$$

Suppose a Lagrangian density of the form

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi + \frac{\lambda}{4!}\varphi^4 + \frac{g}{6!}\varphi^6 \qquad \text{------}(149)$$

* Mass dimension of $\mathcal{L}$:

$$S = \int d^4x\, \mathcal{L} \rightarrow d^4x \rightarrow L^4 = M^{-4} \qquad [d^4x] = -4$$

$$\Rightarrow [\mathcal{L}] = 4 \leftarrow n \qquad \text{------}(150)$$

* Mass dimension of φ:

$$\int d^4x\, m^2\varphi^2$$

$$(\text{mass})^0 = (\text{mass})^{-4}(\text{mass})^2\,[\varphi]^2$$

$$[\varphi] = 1 \leftarrow n \qquad \text{------}(151)$$

* Mass dimension of $\partial\varphi$:

$$\frac{\partial}{\partial x}\varphi \sim \frac{1}{L}\varphi \sim M\varphi \rightarrow [\partial\varphi] = 2 \qquad \text{--}(152)$$

$$\mathcal{L}_{EFT} = A_0 + \overset{\varphi\to-\varphi}{\cancel{A_1\varphi}} + A_2\varphi^2 + \overset{\varphi\to-\varphi}{\cancel{A_3\varphi^3}} + A_4\varphi^4 + A_6\varphi^6 + \cancel{D_1\partial\varphi} + D_2(\partial\varphi)^2 + D_3\varphi\cancel{(\partial\varphi)} + D_4\varphi^2(\partial\varphi)^2 + \dots$$

Lorentz sy

$$\text{------}(153)$$

$[A_0] = 4 \rightarrow A_0 = \bar{A}_0 \Lambda^4$

$[A_2] = 2 \rightarrow A_2 = m^2 \bar{A}_2$

$[A_3] = 1 \rightsquigarrow$ breaks $\varphi \to -\varphi$ symmetry $\left\{ \begin{array}{l} \text{if symmetry not} \\ \text{required, it is} \\ \text{allowed, } \varphi, \varphi^5, \ldots \end{array} \right.$

$[A_4] = 0 \rightarrow A_4 = \lambda/4!$

$[A_6] = -2 \rightarrow A_6 = \bar{A}_6/\Lambda^2$

$[D_2] = 4 \rightarrow D_2 = 1 \leftarrow$ correct kinetic energy

$[D_4] = 6 \rightarrow D_4 = \bar{D}_4/\Lambda^2$

one expects terms $\dfrac{1}{\Lambda^k}$, $k > 0$ expected

to be suppressed $\Rightarrow$ less important

We are back to the 1st lecture!

Literature:

1) Zee: Quantum field theory in a nutshell ← Book

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3) Burgess: An introduction to effective field theory
Ann. Rev. Nucl. Part. Sci. 57, 329 (2007)

4) Luke: Quantum field theory I – Lecture notes
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6) Stewart: Effective field theory, MIT open courseware
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