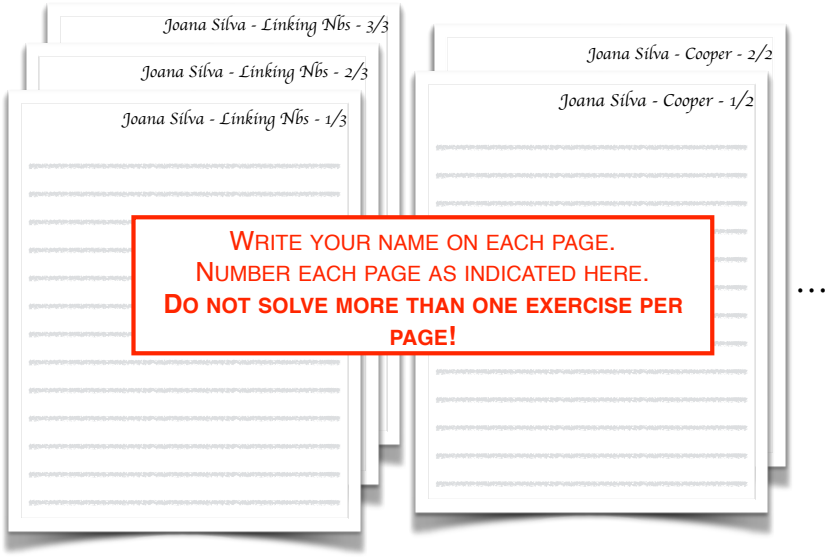
IFT-Perimeter-SAIFR
Journeys into Theoretical Physics 2018
Sunday Exam

Important:



Scores:

- Problem 1 (Cooper Pairing): 20%
- Problem 2 (Effective Field Theory): 20%
- Problem 3 (Constant-w cosmology): 20%
- Problem 4 (Linking Numbers): 20%
- Problem 5 (Biophysics): 20%

◦ Full Name: _____

◦ I am also interested in applying for the Princeton/CUNY program in biological physics: Yes No

◦ I am interested in applying to the IFT masters program even if I am not accepted into the PSI or Princeton/CUNY program: Yes No

◦ If accepted into any of the programs, I would be interested in starting my fellowship at the IFT in August 2018: Yes No

◦ The areas of physics which I am most interested in are:

Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 Cooper Pairing

The ground state of a normal metal can be characterized by a Fermi surface with characteristic wave-vector k_F, such that all states below it are filled and above it are empty. Superconductors can be thought of as an instability of the Fermi surface in presence of an arbitrarily small attractive interaction between electrons.

To understand that, let's go over a simple exercise: consider two extra electrons above the Fermi surface (remember electrons are fermions, so in presence of the Fermi surface, the only "place"one can put electrons is above it since all states below it are already occupied). Considering the Fermi surface to be inert, meaning, its role is simply to block states with $k < k_F$, we can write the Schroedinger equation for the two extra electrons in real space:

$$\left[-\frac{\hbar}{2m}(\nabla_1^2 + \nabla_2^2) + V(\mathbf{r}_1 - \mathbf{r}_2)\right]\Psi(\mathbf{r}_1, \mathbf{r}_2) = E\Psi(\mathbf{r}_1, \mathbf{r}_2), \tag{1}$$

where m is the mass of the electrons, $\hbar$ is Planck's constant, $V(\mathbf{r}_1, \mathbf{r}_2)$ the effective interacting potential between the electrons, and $\Psi(\mathbf{r}_1, \mathbf{r}_2)$ the two-electron wave-function.

1. **[2pt]** Rewrite the problem in terms of relative and center-of-mass (CM) coordinates: $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$.

2. **[4pt]** Separate the CM and relative coordinate parts of the problem: $\Psi(\mathbf{R}, \mathbf{r}) = \Theta(\mathbf{R})\phi(\mathbf{r})$. Assume the CM solution to have the form of a plane-wave: $\Theta(\mathbf{R}) \sim e^{i\mathbf{K}\cdot\mathbf{R}}$ and write the Schroedinger equation for $\phi(\mathbf{r})$ only. What value of $\mathbf{K}$, the CM momentum, minimizes the energy E? Assume this value from here on.

3. **[4pt]** Fourier transform the problem to momentum space using:

$$\phi(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^3} g(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}\,, \qquad \text{and} \qquad V(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^3} v(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{2}$$

   Write the respective Schroedinger equation for $g(\mathbf{k})$. Simplify it as follows: note that the kinetic term and the term carrying the eigenenergy have one integral over momentum, while the potential term has two integrals over momenta. Relabel the variables in the potential term such that you can get rid of one of the two integrals over momenta by factoring it out of the equation. This should give you a self-consistent equation for $g(\mathbf{k})$.

4. **[4pt]**You should have one remaining integral over momentum in the potential term of the self-consistent equation for $g(\mathbf{k})$ you found above.

   Replace the integral over momentum by an integral over energy using the notion of density of states (DOS). Assume the effective potential to be attractive within a narrow shell around the Fermi surface (energies for which the electron-phonon interaction leads to an effective attractive interaction):

$$v(\mathbf{k}) = \begin{cases} -v_0, & \text{for } 0 < \epsilon < \omega_C \\ 0, & \text{otherwise} \end{cases}, \tag{3}$$

where ω_C is a cutoff energy. Here we take the zero of energy to lie at the Fermi energy. Also, assume the DOS to be constant, $N(\epsilon) = N(0)$ within the range of energy $0 < \epsilon < \omega_C$. Note now that inside this integral the dependency of $g(\mathbf{k})$ on momenta can be replaced by a dependency on energy $g(\epsilon)$. Rewrite the self-consistent equation for $g(\mathbf{k})$ in the form:

$$g(\mathbf{k}) = F(\mathbf{k}) \int d\epsilon' g(\epsilon'), \tag{4}$$

making explicit $F(\mathbf{k})$ above.

5. [**4pt**] Perform an extra integral over $\mathbf{k}$ in order to eliminate $g(\mathbf{k})$ from the expression above. Again, write the integral over momentum as an integral over energy. You can now find an expression for the eigenvalue:

$$E = -\omega_C e^{-1/(v_0 N(0))}, \tag{5}$$

where we assumed $v_0 N(0) \ll 1$. This means that the two electrons initially on top of a Fermi surface form a state with negative energy (a bound state) in presence of an arbitrarily small attractive interaction.

6. [**2pt**] Explain how this bound state of fermions can have energy smaller than the Fermi energy (here taken to be zero) in face of the Pauli exclusion principle?

Some useful information about the normal metallic state: Assuming N free electrons in a volume $V = L^3$ with dispersion $\epsilon_{\mathbf{k}} = \hbar^2 \mathbf{k}^2/(2m)$. Here $\mathbf{k} = (k_x, k_y, k_z)$ can only assume quantized values $k_i = n_i 2\pi/L$, where n_i is an integer and $i = x, y, z$, so we can write $\mathbf{k} = (n_x, n_y, n_z)2\pi/L$. We can write the total number of electrons as equal the total number of occupied states in momentum space (states with $k < k_F$):

$$N = \sum_{\mathbf{k}, k<k_F} 1 = \sum_{\mathbf{k}, k<k_F} \Pi_{i=x,y,z} \frac{\Delta k_i L}{2\pi} = V \sum_{\mathbf{k}, k<k_F} \frac{\Delta \mathbf{k}}{(2\pi)^3}, \tag{6}$$

so the density of electrons can be given by:

$$n = \frac{N}{V} = \sum_{\mathbf{k}, k<k_F} \frac{\Delta \mathbf{k}}{(2\pi)^3} \xrightarrow{Continuum} \int_{\mathbf{k}, k<k_F} \frac{d\mathbf{k}}{(2\pi)^3}, \tag{7}$$

Given the dispersion relation, we can change variables in the integral from momentum to energy:

$$n = \int_0^{\epsilon_F} \frac{k^2 dk d\Omega}{(2\pi)^3} = \int_0^{\epsilon_F} \frac{k^2}{2\pi^2} \frac{dk}{d\epsilon} d\epsilon = \int_0^{\epsilon_F} \frac{m^{3/2} \epsilon^{1/2}}{\pi^2 \sqrt{2}} d\epsilon = \int_0^{\epsilon_F} N(\epsilon) d\epsilon \tag{8}$$

where $\epsilon_F = \hbar^2 k_F^2/(2m)$ and $N(\epsilon)$ is the density of states (number of states in the interval of energy $d\epsilon$ around ϵ).

2 Effective Field Theory

There are two basic uses for effective field theories: (1) top-down and (2) bottom-up.

1. **[2pt]** Explain what are the differences between the two. Provide an example for each of them.

Invariably, one can identify three steps in the construction of an effective field theory to describe a system in which there is a large hierarchy of scales: (1) determination of the relevant degrees of freedom, (2) identification of symmetries, (3) use of a power counting scheme.

2. **[2pt]** Explain the meaning of each of these steps. If you wish, you can use an example to substantiate your explanation.

Once one has settled on the above three steps, one writes down a Lagrangian to compute, for example, a correlation function of the field variables in the Lagrangian that can be related to observables, as e.g. a scattering amplitude. In the calculation process of a correlation function, in general one encounters ultraviolet divergences and one needs to employ a regularization procedure and then renormalize the theory to obtain meaningful results.

3. **[2pt]** Explain the generic physical origin of such ultraviolet divergences. Explain the meaning of regularization and renormalization.

In the following, let us consider an example of a top-down use of an effective field theory to simplify a calculation in a specific situation. Suppose that the classical Lagrangian density of "a fundamental theory" is given by (consider natural units, $\hbar = 1$ and $c = 1$):

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) + \frac{1}{2}\left(\partial_\mu\Phi\partial^\mu\Phi - M^2\Phi^2\right) - \frac{1}{4!}g\phi^4 - \frac{1}{2}\lambda\phi^2\Phi \tag{9}$$

where ϕ and Φ are real scalar fields. Please ignore the fact that this Lagrangian leads to a vacuum energy that is unbounded from below. The tree-level amplitude of the $\phi(p_1) + \phi(p_2) \to \phi(p_3) + \phi(p_4)$ scattering can be written as

$$\mathcal{A}(p_1, p_2; p_3, p_4) \;=\; -g - \lambda^2\left(\frac{1}{s - M^2 + i\epsilon} + \frac{1}{t - M^2 + i\epsilon} + \frac{1}{u - M^2 + i\epsilon}\right) \tag{10}$$

where p_1 and p_2 (p_3 and p_4) are the initial (final) four-momenta of the $\phi-$particles, and $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$.

4. **[2pt]** Determine the mass dimensions of the fields ϕ and Φ and of the couplings g and λ.

5. [**5pt**] Consider $M \gg m$. Show that the on-shell scattering amplitude up to $\mathcal{O}(1/M^4)$ is given by

$$\mathcal{A}_{\text{on-shell}} = -g + \lambda^2 \left(\frac{3}{M^2} + \frac{4m^2}{M^4} \right) \tag{11}$$

Recall that the mass-shell condition for a particle of mass m with four-momentum p is $p^2 = m^2$.

6. [**5pt**] Show that the result (11) can be obtained with the effective Lagrangian density involving solely the ϕ field:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) - \frac{1}{4!} g_{\text{eff}} \phi^4 \tag{12}$$

with the effective coupling g_{eff} given by

$$g_{\text{eff}} = g - \lambda^2 \left(\frac{3}{M^2} + \frac{4m^2}{M^4} \right) \tag{13}$$

7. [**2pt**] Explain what has been done in this simple example with respect to the effective field theory terminology of *hierarchy of scales*, *integrating out large energy scales*, and *matching*.

3 Constant-w cosmology

In this problem we'll study a constant-w cosmology, i.e. a universe whose stress-energy is dominated by a single species of matter whose equation of state $w = p/\rho$ is independent of time. In this case, there are only two parameters: w and H_0, where H_0 is the Hubble expansion rate today (i.e. at $a = 1$).

For simplicity, assume that $w > -1$. (This ensures that the elapsed proper time between $a = 0$ and $a = 1$ is finite, i.e. the universe begins in a big bang.) We use units where $c = 1$.

1. **[3pt]** Derive an expression for the Hubble parameter $H(a)$ as a function of scale factor a (and the parameters H_0, w).

2. **[3pt]** Using your answer to the previous part, integrate Friedmann's equation to solve for the scale factor a as a function of proper time t. Show that the time t since the big bang is related to the instantaneous Hubble expansion rate H by:

$$t = \frac{2}{3(1+w)} H^{-1} \tag{14}$$

   at all times during the expansion.

3. **[3pt]** Suppose a rocket is launched today (i.e. at $a = 1$) with initial velocity $v_0 < 1$, and subsequently does not accelerate. Show that its velocity at all subsequent times is:

$$v = \frac{v_0}{\sqrt{v_0^2 + (1 - v_0^2)a^2}} \tag{15}$$

4. **[3pt]** Write an integral expression for the total *comoving* distance $(\Delta x)_c$ traveled by the rocket between $a = 1$ and $a = \infty$. For which values of w is $(\Delta x)_c$ finite?

5. **[8pt]** For arbitrary w, the integral in the previous part can't be done analytically, so let's expand $(\Delta x)_c$ as a series in v_0. Assuming that w is chosen so that $(\Delta x)_c$ is finite, show that the first two terms in this expansion are:

$$(\Delta x)_c = \frac{v_0}{H_0} \left(\frac{2}{1 - 3w} \right) \left[1 + \frac{2}{5 - 3w} v_0^2 + \cdots \right] \tag{16}$$

4 Linking Numbers

A knot is a closed curve in $\mathbb{R}^3$, parameterized by a function $f : S^1 \to \mathbb{R}^3$, $f(\theta) = (f_1(\theta), f_2(\theta), f_3(\theta))$, $f_i(\theta + 2\pi) = f_i(\theta)$. We assume that $f(\theta), f(\theta')$ are different unless $\theta - \theta' = 2\pi N$ for an integer N.

Two knots which don't touch each other form a link. Suppose we have a link given by the knots $f(\theta)$, $g(\theta)$. The condition that they don't touch each other means $f(\theta)$, $g(\theta')$ are always different.

Define a map

$$F : S^1 \times S^1 \to \mathbb{R}^3 \setminus 0$$
$$F(\theta_1, \theta_2) = f(\theta_1) - g(\theta_2).$$

Let

$$\omega = r^{-3} \left(x_1 \mathrm{d}x_2 \wedge \mathrm{d}x_3 - x_2 \mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \right)$$

be the two-form on $\mathbb{R}^3 \setminus 0$ we discussed in class.

Define the *Gauss linking number* by

$$L(f, g) = \int_{S^1 \times S^1} F^* \omega.$$

1. **[5pt]** Give an explicit formula for $F^*\omega$ as a form on $S^1 \times S^1$, in terms of the parameterized knots f, g.

2. **[5pt]** Suppose we change the knots f, g in such a way that the link changes continuously, i.e. without passing the strands through each other. Explain, using the results derived in class, why the linking number does not change.

3. **[10pt]** Calculate the linking number of the linked circles given by

$$f(\theta) = (\cos\theta, \sin\theta, 0)$$
$$g(\theta') = (0, 1 + \cos\theta', \sin\theta')$$

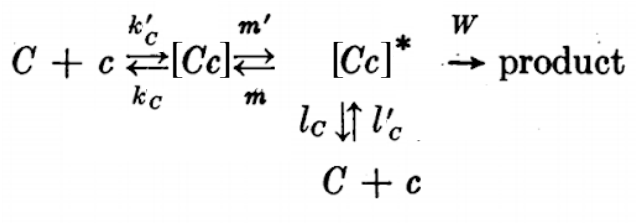
   It is possible to do this directly. But, it helps to use the fact that the answer doesn't change if we deform the two circles without letting them cross. One way to do this is as follows (there are others). We can deform so that the circle parameterized by $f(\theta)$ has some very small radius ε, and is still centered at the origin. We can stretch the circle parameterized by $g(\theta')$ so that near the origin ($\theta' = \pi$) it looks a very long straight line in the z-axis, and the rest of this circle is very far away form the origin.

   To solve this problem, it might help to know that

$$\int_{s=-\infty}^{\infty} \mathrm{d}s \frac{1}{(1 + s^2)^{3/2}} = 2.$$

5 Biophysics

1. Consider the kinetic proofreading scheme proposed by Hopfield where c and C are two elements which can bind together to form either $[Cc]$ or $[Cc]^*$ according to the diagram



   (a) **[2pt]** In terms of the concentrations of c, C, $[Cc]$ and $[Cc]^*$, and the rates k'_c and k_c, l'_c and l_c, m' and m, and W, first write the equation that implies that the concentration of c is constant.

   (b) **[1pt]** Secondly, write the equation that implies that the concentration of $[Cc]$ is constant.

   (c) **[1pt]** Thirdly, write the equation that implies that the concentration of $[Cc]^*$ is constant.

   (d) **[1pt]** Argue that these 3 equations can be used to determine the steady-state concentration of $[Cc]^*$.

   (e) **[3pt]** Suppose there is an incorrect process where c is replaced with d in all of the processes. How would you use this information, together with the corresponding concentration for the incorrect pair $[dC]^*$, to determine the error fraction?

2. **[4pt]** Determine the maximum entropy distribution where only the mean activity of each neuron is constrained to be equal to its observed value m_i

3. We defined the mutual information to be the Kullback-Leibler divergence between the joint distribution of two variables and the product of their marginals, $I(X;Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$. Starting from this definition, show that the mutual information can also be expressed as:

   (a) **[4pt]** $I(X;Y) = H(Y) - H(Y|X)$.

   (b) **[4pt]** $I(X;Y) = H(X) + H(Y) - H(X,Y)$.

   Some useful definitions from class:

   Given a joint distribution $p(x,y)$, we can define the entropy of this joint distribution to be $H(X,Y) = -\sum_{x,y} p(x,y) \log p(x,y)$. Next, define the marginal distributions, $p(x) = \sum_y p(x,y)$ and $p(y) = \sum_x p(x,y)$. These distributions have corresponding entropies, $H(X) = -\sum_x p(x) \log p(x)$, and $H(Y) = -\sum_y p(y) \log p(y)$. Finally, recall the definition of conditional entropy, $H(Y|X) = -\sum_x p(x) \sum_y p(y|x) \log p(y|x)$.