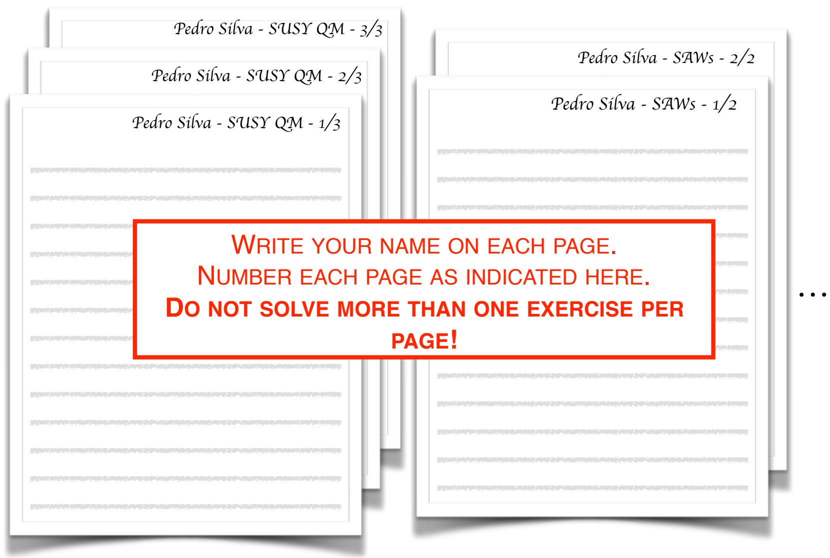
IFT-Perimeter-SAIFR

Journeys into Theoretical Physics 2018

Saturday Exam



Scores:

- Problem 1 (Self Avoiding Walks): 25%
- Problem 2 (Special Relativity with a Compact Direction): 25%
- Problem 3 (Angular Momentum Paradox): 25%
- Problem 4 (SUSY Quantum Mechanics): 25%

∘ Full Name: ____________________

∘ I am also interested in applying for the Princeton/CUNY program in biological physics: [Yes] [No]

∘ I am interested in applying to the IFT masters program even if I am not accepted into the PSI or Princeton/CUNY program: [Yes] [No]

∘ If accepted into any of the programs, I would be interested in starting my fellowship at the IFT in August 2018: [Yes] [No]

∘ The areas of physics which I am most interested in are:

____

Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 Self Avoiding Walks (SAWs)

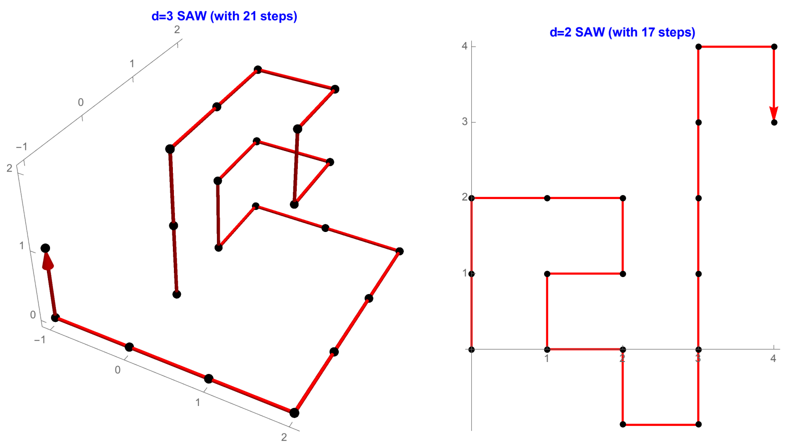


Figure 1: Self Avoiding Walks are paths with n steps on a d dimensional lattice starting at the origin. At each step connects o point to one of its $2d$ neighbours. The key constraint defining the self-avoiding paths is that no point is visited twice.

SAW are paths starting at the origin and moving in a lattice (here taken to be a cubic lattice) without ever visiting the same site twice as illustrated in figure 1. Such objects turn out to be relevant for the study of long polymers whose roughly random shape resembles that of a SAW. In this exercise we will study some of the properties of the statistics of such paths.

Some inequalities

Let $c_n^{(d)}$ be the number of SAW of length n in a d-dimensional cubic lattice as represented in figure 1. ($c_0^{(d)} = 1$ for the empty path.)

1. **[3pt]** Show that
$$d^n < c_n^{(d)} \leq 2d(2d-1)^{n-1}\,. \tag{1}$$

2. **[3pt]** Show that
$$c_{n+m}^{(d)} \leq c_n^{(d)} c_m^{(d)} \tag{2}$$

Never Look Back SAWs (NLB-SAWs)

In this section we consider a subset of SAW called the NLB-SAWs which are $2d$ SAW in a semi-infinite $\{0,1\} \times \mathbb{N}$ strip which never move to the left, see figure 2. Let c_n be the number of such paths of length n.

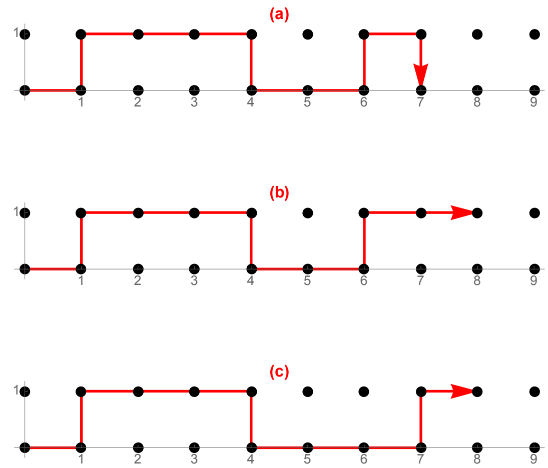


Figure 2: SAW starting on the origin and moving inside the semi-infinite strip $\{0,1\} \times \{0,1,2,3,\dots\}$ with the key constraint that the paths never move to the left. These are called *Never Look Back* SAW or NLB-SAW. In a given path, the last step can be vertical (as in the top figure) or horizontal (as in the middle and bottom examples).

3. **[3pt]** Let v_n be the number of paths whose last step is vertical and h_n be the number of paths whose last step is horizontal. Show that

$$h_{n+1} = v_n + h_n\,, \qquad v_{n+1} = h_n\,. \tag{3}$$

   Hint: Note the three examples in figure 2.

4. **[3pt]** Show that c_n obeys a simple recursion relation and work out the first few c_n's up to $n = 5$. (You should find $c_4 = 8$.)

5. **[3pt]** Show that

$$c_n = A_+\lambda_+^n + A_-\lambda_-^n \tag{4}$$

   and find $A_\pm$ and $\lambda_\pm$. How does c_n behave at large n?

   Hint: $\lambda_+ = \frac{1+\sqrt{5}}{2}$.

NLB-SAW with gravity

We now consider a toy model where we study the exact same paths as in the previous section with the extra addition of gravity. More precisely, we consider the partition function

$$Z = \sum_{\text{NLB-SAWs}} e^{-\beta g H - \beta \mu n} \tag{5}$$

where n is the number of steps of a the *Never Look Back* SAW in the strip and H is the number of sites in the upper row for that path (In figure 2, for instance, we have $n = 11$

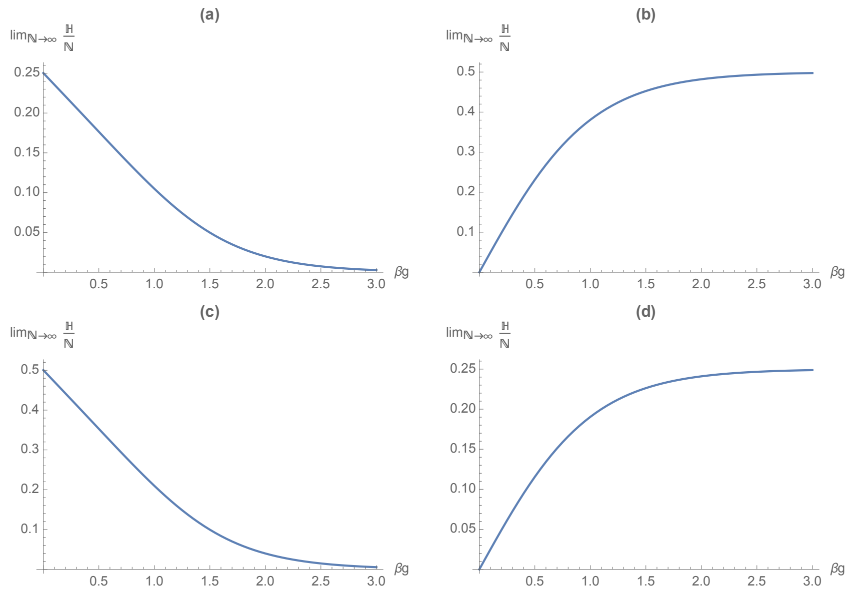


Figure 3: Fraction of time the NLB-SAW paths spend on the top row as a function of the gravity parameters. Only one of these plots is the correct one, see problem 7.

for all examples and $H = 6, 7, 6$ for the top, middle and bottom NLB-SAW respectively.) The gravitational constant g, the inverse temperature β and the chemical potential μ are all constants. In the last point we discuss the computation of the partition function. For now you can use the result

$$Z = \frac{e^{\beta\mu} + e^{4\beta\mu+2\beta g}}{e^{2\beta\mu+\beta g} - e^{3\beta\mu+\beta g} - e^{3\beta\mu+2\beta g} + e^{4\beta\mu+2\beta g} - 1} \,. \tag{6}$$

6. **[3pt]** Compute the average length and average height as a function of the gravity constant g, the chemical potential μ and the inverse temperature β,

$$\mathbb{N} = \frac{1}{Z} \sum_{\text{NLB-SAWs}} n\, e^{-\beta g H - \beta\mu n}\,, \qquad \mathbb{H} \equiv \frac{1}{Z} \sum_{\text{NLB-SAWs}} H\, e^{-\beta g H - \beta\mu n} \tag{7}$$

   Hint: The expressions are kind of ugly, no need to try to simplify them.

7. **[3pt]** From the previous results we can eliminate μ to get $\mathbb{H}(\mathbb{N}, \beta g)$. (In practice this is hard to do analytically.) We can then study the ratio of points in the upper row to the total number of points for very large polymers, $r \equiv \lim_{\mathbb{N}\to\infty} \mathbb{H}/\mathbb{N}$.

   What do you expect to obtain for very small gravity $\beta g \ll 1$ or very large gravity $\beta g \gg 1$? One of the plots in figure 3 is the correct plot for this ratio; which one?

8. **[4pt]** Explain what would your strategy be to establish (6). (If you derive this expression even better of course but since the computation is somehow non-trivial, an explanation of the steps involved is enough.)

2 Special Relativity with a Compact Dimension

Suppose we live in a universe where one of the 3 spatial dimensions has a large finite length L and is periodic. In other words, the spacetime point (t, x, y, z) is identified with the spacetime point $(t, x + nL, y, z)$ for any integer n.

1. Suppose there was a supernova explosion at time $t = t_0$ and at position $(x, y, z) = (x_0, y_0, 0)$ in the reference frame of a stationary observer on Earth who is located at the position $(x, y, z) = (0, 0, 0)$. At the time of the explosion, the star was moving at velocity v in the y direction.

   (a) [**3pt**] When will the observer on Earth see the supernova explosion?

   Hint: He will receive multiple signals of the explosion.

   (b) [**6pt**] If the supernova explosion emits light of frequency f, find the frequency of the light observed on Earth for the various signals received

   Hint: You can check that for $x_0 = 0$ your formula reduces to the familiar Doppler shift.

2. We will now consider the twin paradox in this universe with one compact dimension. One of the twins is on Earth and the other twin is flying on a rocket with constant velocity v in the x direction. So from the point of view of the twin in the rocket, the twin on Earth is flying with a constant velocity $-v$. The twins synchronize their watches to $t = t' = 0$ when the rocket flies by Earth. Each time the rocket makes one trip around the universe, it will fly by Earth and the twins can compare their ages.

   (a) [**5pt**] Suppose the twin in the rocket ship emits a signal of frequency f at time t in his reference frame. When is the signal observed by the twin on Earth and what is the observed frequency?

   (b) [**3pt**] Suppose the twin on Earth emits a signal of frequency f' at time t' in his reference frame. When is the signal observed by the twin on the rocket and what is its observed frequency?

   (c) [**5pt**] When the twins cross paths after the rocket makes one trip around the universe, what (if any) is the difference in their ages?

   (d) [**3pt**] Compare with the usual twin paradox where one twin stays on Earth and the other twin departs on a rocket first with velocity v and then returns with velocity $-v$. Explain the reason for any different conclusions between the usual twin paradox and this new situation.

3 Angular Momentum Paradox

Note that there are some potential useful formulae at the bottom of this exercise.

Before we start the problem proper, let's deduce a relevant quantity:

1. **[4pt]** Show that the density of momentum carried by electromagnetic fields in vacuum is given by:
$$\vec{\tilde{p}} = \frac{\vec{E} \times \vec{B}}{\mu_0 c^2} \tag{8}$$
and keep in mind that the density of angular momentum will be given by $\vec{\tilde{L}} = \vec{r} \times \vec{\tilde{p}}$.

Now, consider the following setup: two coaxial nonconducting cylindrical shells with very long lengths l. The smaller shell, "cylinder A", has radius a and a total uniformly distributed charge Q. The bigger shell, "cylinder B", has radius b and charge $-Q$ (also uniformly distributed). These two cylinders are free to rotate around their axes. They are inside a equally long and coaxial solenoid of radius R ($R > b > a$) which is carrying a constant current I, generating a constant magnetic field B_0 in the region of the cylinders. Both cylinders are initially at rest. From this initial static setup, imagine the

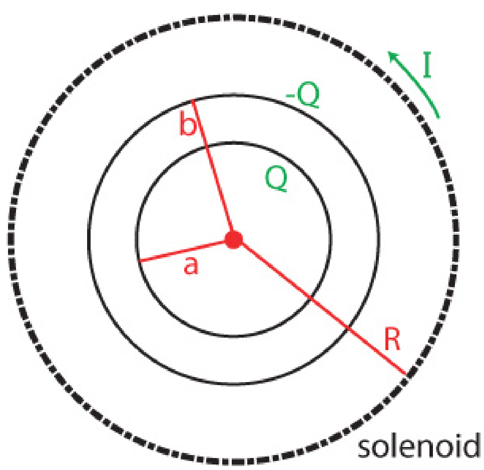


Figure 4: transversal cut of the setup described in the text

current on the solenoid is decreased to zero (without any external force applied to the system, e.g. the solenoid is a superconductor slowly heating up, and suddenly becomes a normal conductor above some critical temperature, which then starts to kill the current by resistance).

2. **[4pt]** Show that the instantaneous eletric field induced by the changing field $\vec{B}$ at radius r is given by:
$$\vec{E} = \frac{r}{2} \left| \frac{dB}{dt} \right| \hat{\varphi}, \tag{9}$$
where $B = |\vec{B}|$ and $\hat{\varphi}$ is the counterclockwise direction in figure 4.

3. **[4pt]** Find the angular momentum gained by each cylinder by the end, when the solenoid magnetic field has decreased to zero. (Assume that the two cylinders are rotating slowly enough that you can completely disregard the magnetic fields generated by them).

4. **[4pt]** Calculate the electric field in all regions of space in the initial static situation.

5. **[5pt]** Is angular momentum conserved? If so, show it quantitatively.

6. **[4pt]** Now imagine we repeat the experiment without "cylinder B". Discuss angular momentum conservation in this case.

Useful formulae:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{(Faraday's Law)} \tag{10}$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 J \quad \text{(Ampère's Law)} \tag{11}$$

$$\nabla . \vec{E} = \frac{\rho}{\epsilon_0} \quad \text{(Gauss's Law)} \tag{12}$$

$$\nabla . \vec{B} = 0 \tag{13}$$

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \quad \text{(Poynting Vector)} \tag{14}$$

4 Supersymmetric Quantum Mechanics

Consider a spin 1/2 particle moving in a line in the presence of some external potential $V(x)$ and of some magnetic $B(x)$. The Schrödinger equation with the Hamiltonian

$$H = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\mathbb{I} + B(x)\sigma_3\,, \tag{15}$$

has two components. (In what follows we set $\hbar = 2m = 1$ to simply the analysis.) Furthermore we take

$$V(x) = \left(\frac{dW}{dx}\right)^2\,, \qquad B(x) = \frac{d^2W}{dx^2}\,. \tag{16}$$

As we will see with this choice this model becomes quite special: it is supersymmetric. Studying it will also allow us to unveil remarkable relations between otherwise unrelated Schrödinger problems of the form

$$-\psi''(x) + V(x)\psi(x) = E\,\psi(x)\,. \tag{17}$$

We use the following representation of the Pauli matrices:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,. \tag{18}$$

1. **[2pt]** Show that $[H, \sigma_3] = 0$.

This means we can assign a definite spin in the z direction to each eigenstate. In other words, whatever results we derive will apply to the two bosonic Schrödinger problems given by the top and bottom component of the spin 1/2 equation $H\psi = E\psi$ where ψ is a two-component object and the Hamiltonian is given by (15).
Consider next the operators

$$Q = \frac{1}{i}\left(\frac{d}{dx} + \frac{dW}{dx}\right)\sigma_+ \qquad \text{and} \qquad Q^\dagger = \frac{1}{i}\left(\frac{d}{dx} - \frac{dW}{dx}\right)\sigma_-\,. \tag{19}$$

They obey the so-called $su(1|1)$ super-algebra given by

$$\{Q, Q^\dagger\} = H\,, \qquad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0\,, \qquad [Q, H] = [Q^\dagger, H] = 0\,. \tag{20}$$

where $\{A, B\} \equiv AB + BA$ is called the *anti-commutator* and $[A, B] = AB - BA$ is the usual commutator.

2. **[2pt]** Derive one of the relations in (20).

3. **[2pt]** Show that all energies $E_n \geq 0$.

4. **[2pt]** Show that the ground state energy vanishes $E_0 = 0$ if and only if $Q\psi_0 = Q^\dagger\psi_0 = 0$.

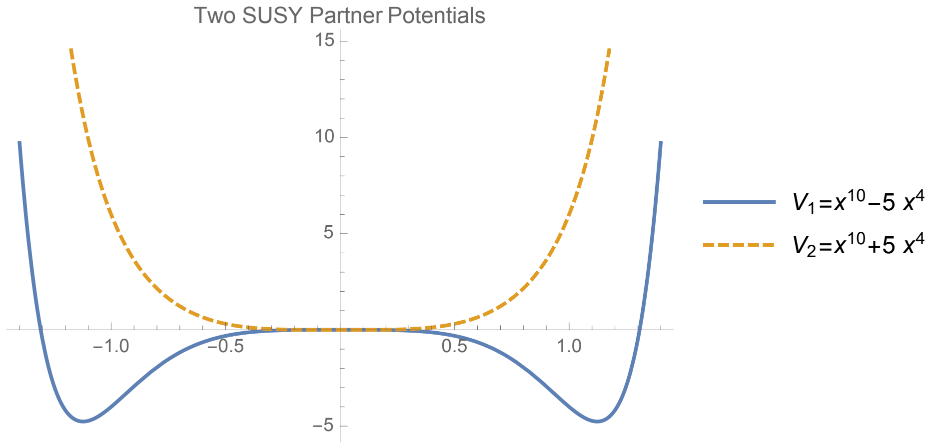


Figure 5: Although the potentials V_1 and V_2 look very different - one is a simple well while the other is a double well - their spectrum is related in a very simple way as a consequence of problem 6. Namely, the energy $E^{(1)}_{n+1}$ of any excited state $\psi^{(1)}_{n+1}$ for potential V_1 is the same as the energy $E^{(2)}_n$ of the state $\psi^{(2)}_n$ for the problem with potential V_2. That is $E^{(1)}_{n+1} = E^{(2)}_n$ except for the ground state energy of V_1 which is unpaired.

5. **[2pt]** Show that these conditions lead to

$$\psi_0 = \begin{pmatrix} A\, e^{W(x)} \\ B\, e^{-W(x)} \end{pmatrix} \tag{21}$$

where A and B are constants. Suppose $W(x) \to \infty$ as $x \to \pm\infty$. What should we take for these constants?

6. **[3pt]** Consider now excited states. Show that they come in degenerate spin-up/spin-down states related by the action of Q.

7. **[2pt]** As an application, we can now relate seemingly unrelated standard bosonic Schrödinger problems of the form (17): one with $V(x)$ given by $V_1(x) = x^{10} - 5x^4$, and another with a potential $V_2(x) = x^{10} + 5x^4$, see figure 5

   Show that these potentials are such that the two problems can be thought of as two components of a same SUSY problem (15), (16).

   Potentials V_1 and V_2 with such property are SUSY partner potentials.

8. A pair of SUSY partner potentials V_1, V_2 is shape invariant if

$$V_2(x; a_1) = V_1(x; a_2) + R(a_1) \tag{22}$$

where a_1 is a parameter, $a_2 = f(a_1)$, and $R(a_1)$ is independent of x. The Schrödinger problem for V_1 has a ground state with zero energy. Shape invariance can be used to determine the exact spectrum in many cases.

(a) **[2pt]** By constructing a sequence of shape invariant SUSY partner potentials show that the nth energy eigenvalue for $n > 0$ for the potential V_1 is

$$E_n^{(1)} = \sum_{k=1}^{n} R(a_k) \tag{23}$$

where $a_{k+1} = f(a_k)$.

(b) **[2pt]** Show that the potentials

$$V_1(x; B) = B^2 - B(B+1)\text{sech}^2 x \tag{24}$$

and

$$V_2(x; B) = B^2 - B(B-1)\text{sech}^2 x \tag{25}$$

are SUSY partners.

(c) **[2pt]** Show that the potentials

$$V_1(x; B) = B^2 - B(B+1)\text{sech}^2 x \tag{26}$$

and

$$V_2(x; B) = B^2 - B(B-1)\text{sech}^2 x \tag{27}$$

are shape invariant.

(d) **[2pt]** Determine the spectrum of $V_1(x; B) = B^2 - B(B+1)\text{sech}^2 x$.

(e) **[2pt]** How many bound states does $V_1(x; B) = B^2 - B(B+1)\text{sech}^2 x$ have?