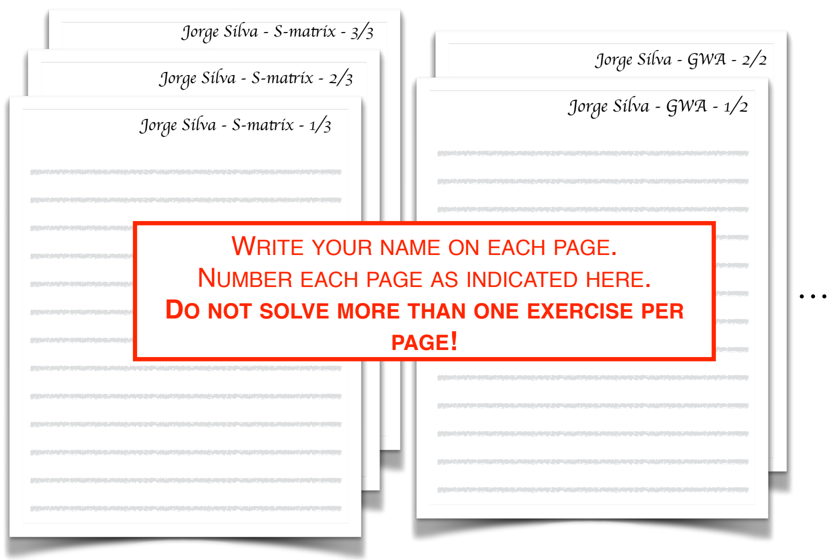
IFT-Perimeter-SAIFR

Journeys into Theoretical Physics 2022

Sunday Exam

• Write your name on each page • Number each page as indicated • Do not solve more than one exercise per page.



- Problem 1 (S-matrix Bootstrap): 25%
- Problem 2 (Earth as a Detector): 25%
- Problem 3 (Gravitational Wave Astronomy): 25%
- Problem 4 (Quantum phase transition in the 1d rotor model): 25%

Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 S-matrix Bootstrap

Two bound-states in an ultra-local theory

Consider a unitarity ultra-local theory with only two bound-states. One is a single pole of $S(k)$ at $k = i\kappa$ with residue $i \times g^2$ and the other is a single pole of $S(k)$ at $k = i\kappa'$ with residue $i \times g'^2$. The allowed combined space for g^2 and g'^2 is plotted in this figure:

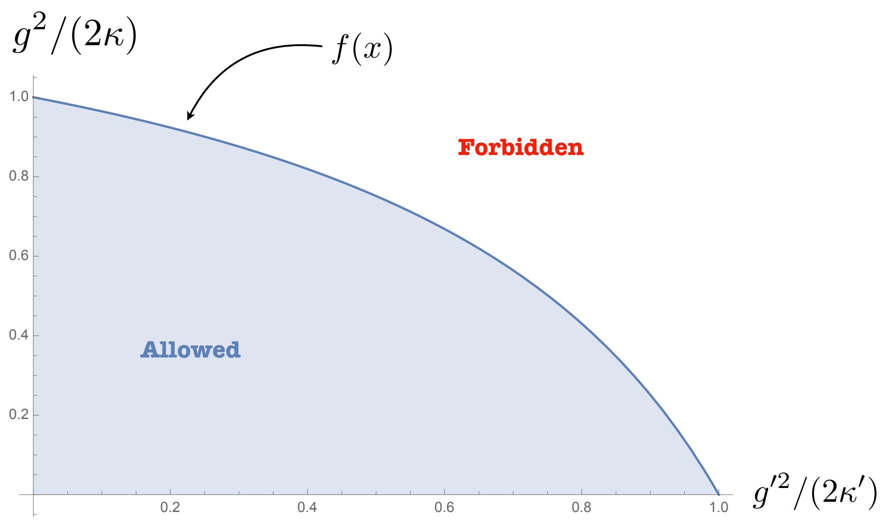


Figure 1: The curve $f(x)$ with $f(0) = 1$ and $f(1) = 0$ determines the boundary between allowed and forbidden couplings. This function depends on $x = g'^2/(2\kappa')$ and also on the ratio $\alpha = \kappa'/\kappa$. In this plot $\alpha = 10$.

The optimal S-matrices living at the boundary between the forbidden and allowed space are given by the same ansatz found in the lecture (a product of three factors: a zero and two poles) which we used to produce the absolute values $g^2 \leq 2\kappa$ and $g'^2 \leq 2\kappa'$ in the lectures and leading to $f(0) = 1$ and $f(1) = 0$.

1. **[10pt]** Find $f(x)$.

   Hint: You should find $f(x) = (A+Bx)/(C+Dx)$ where $A, B, \ldots$ depend on $\alpha = \kappa'/\kappa$.

One bound-state in a finite range theory

2. **[10pt]** Consider a theory with finite range and a single bound-state at $k = i\kappa$. We do not know the precise range of the potential but we do know it is finite range. The coupling to that bound-state $g^2 = -i\,\mathrm{res}_{k=i\kappa} S(k)$ was measured experimentally to be $g^2 \simeq 3\kappa$. Estimate the minimum range the potential must have.

Gaussian potential

3. **[5pt]** What singularities do you expect in the upper half plane for $S(k)$ for a potential $V(x) = \mu/a \exp(-x^2/a^2)$ for a small? What about for larger a?

2 Earth as a detector

Consider a dark matter candidate ϕ which couples to the standard model like

$$\mathcal{L} \supset \frac{\phi}{F_\phi} m_e \bar{e} e \tag{1}$$

in addition to the Standard Model Lagrangian (You are allowed to be a $\pi = 3 = 1$ physicist like me from questions 3, and answers only need to be right on the order of magnitude level).

1. **[3pt]** Write down how the effective electron mass oscillates if a background ϕ field oscillates as $\phi(t) = A_\phi \cos m_\phi t$.

2. The electron mass is a fundamental parameter that is very important for our everyday life. How would these quantities change if we increase only the electron mass by 10%? (Your answer can be increase/decrease by X percent, increase/decrease by much less than a percent, does not change at all)

   (a) **[2pt]** Size of an atom

   (b) **[2pt]** Frequency of an hydrogen spectral line (Balmer series)

   (c) **[2pt]** Surface area of a football

   (d) **[2pt]** The opening angle of water and CO_2 molecules

   (e) **[2pt]** The mass of a human

3. **[4pt]** The radius of the earth oscillates as a result of the oscillating ϕ field and the scalar wave is transferred into a sound wave. How does the radius of the earth oscillate in the background oscillating $\phi(t)$ field

4. **[4pt]** The earth has a radius of 6400km and a sound velocity of 10^3 m/s, what is the approximate frequency of the lowest resonant mode of the earth (in Hz)? What mass of dark matter would this resonant mode be sensitive to?

5. **[4pt]** The earth vibration can be measured to a fractional sensitivity of 10^{-17}, what does this imply for the sensitivity on A_ϕ and F_ϕ? How does F_ϕ compare to the Planck scale?

(See supplemental material in the next page for the Standard Model Lagrangian, useful natural unit conversion and other important physical quantities you might need)

Supplemental materials

Units and constants

Here are some useful constants and units you might need.

Speed of light: $c = 3 \times 10^8 m/s$
Planck scale: $m_{\text{pl}} = 2.4 * 10^{18}\,\text{GeV}$.
Mass of proton: 0.938 GeV
Local dark matter density: $\rho = 0.3\,\text{GeV/cm}^3$.
$\text{GeV} * \text{cm} = 5 * 10^{13}$.
$\text{GeV} = 10^9$ eV.

Standard model Lagrangian at low energies

$$\mathcal{L} \supset -m_e \bar{e} e - m_p \bar{p} p - \frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \cdots \tag{2}$$

3 Gravitational Wave Astronomy

1. Suppose gravitational radiation is quantized, then use the quadrupole formula derived in the course valid for a system with non-relativistic internal velocity in circular orbit to estimate how many gravitons are emitted per orbital period for the following systems:

   (a) [**3pt**] Hydrogen atom

   (b) [**3pt**] Earth-Sun system

   (c) [**3pt**] Macroscopic lab system of 1 kg masses separated by 1 m

2. [**5pt**] Do the above systems emit more or less than 1 graviton quantum per orbit?

3. [**5pt**] For non-gravitationally bound system, deduce a bound on the size and frequency of the system for quantized gravitational radiation to be emitted, assuming that with less than 1 quantum per orbital period no emission is produced.

4. [**6pt**] For earth-sun system estimate the lifetime for gravitational radiation decay. Is it larger or smaller than the age of the Universe?

Useful constants (units $\hbar = c = 1$, use $\hbar \times c \simeq 200 MeV \times fm$, $c = 3.00 \times 10^8 m/sec$ to reintroduce correct dimensions):

- Fine structure constant (e is the electron charge)

$$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}\,, \tag{3}$$

- electron mass

$$m_e \simeq 0.511 MeV\,, \tag{4}$$

- proton mass

$$m_p \simeq 938 MeV\,, \tag{5}$$

- Bohr's radius: $a_0 = (m_e \alpha)^{-1}$
- Earth-Sun distance: 1.50×10^8 km.
- Mass of the Sun $M_\odot \simeq 1.99 \times 10^{33} g$
- Earth's mass: $M_\oplus \simeq 5.97 \times 10^{24} g$
- Hubble constant: $H_0 \simeq 70 km/sec/Mpc$

4 Quantum phase transition in the 1d rotor model

Consider the rotor model in the one-dimensional chain with L sites and the periodic boundary condition,

$$\hat{H} = \frac{U}{4}\sum_{j=1}^{L}\hat{n}_j^2 - 2t\sum_{j=1}^{L}\cos(\hat{\theta}_j - \hat{\theta}_{j+1}), \tag{6}$$

where $\hat{\theta}_{L+1} = \hat{\theta}_1$. $\hat{\theta}_j$ represents the angular variable defined at site j and $\hat{n}_j$ is the angular momentum operator that satisfies $[\hat{\theta}_j, \hat{n}_k] = i\delta_{jk}$.

1. **[2pt]** Prove that $e^{\pm i\hat{\theta}_j}$ acts as the raising and lowering operator of n_j, respectively, by showing that $[\hat{n}_j, e^{\pm i\hat{\theta}_k}] = \pm\delta_{jk}e^{\pm i\hat{\theta}_k}$.
2. **[5pt]** Find the wavefunctions and energies of the ground state and all first excited states for $U > 0$ and $t = 0$.
3. **[10pt]** With a non-zero but small t with $t \ll U$, we can treat the hopping (t) term in the Hamiltonian as a perturbation. Using the perturbation theory, compute the correction to the energies of the ground state and the first excited states to the linear order or t/U.
4. **[3pt]** As t increases, the energy gap between the ground state and the first excited state decreases. When the gap vanishes, we expect that the nature of the ground state changes qualitatively in the $L \to \infty$ limit. Based on the above calculation, estimate the magnitude of t at which such phase transition occurs.
5. **[5pt]** Describe what differentiate the ground state in the small t/U limit and the large t/U limit.