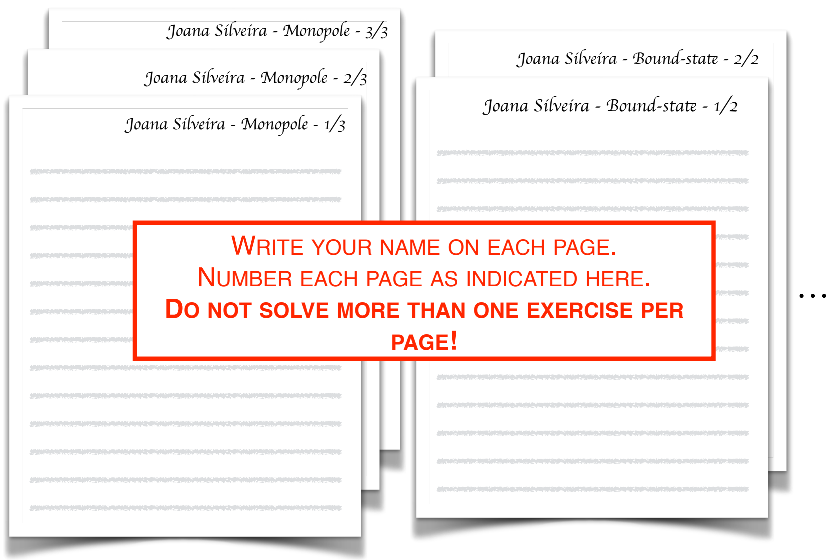
IFT-Perimeter-SAIFR

Journeys into Theoretical Physics 2022

Saturday Exam

• Write your name on each page • Number each page as indicated • Do not solve more than one exercise per page.



- Problem 1 (Speed of Light vs Speed of Gravity): 25%
- Problem 2 (Inevitability of Bound-states): 25%
- Problem 3 (Magnetic Monopole Quantization): 25%
- Problem 4 (Neutron Star): 25%

Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 Speed of Light vs Speed of Gravity

In 2017, the collision of two neutron stars was used to put bounds on alternative theories of general relativity which predict a mass for the graviton. When the two neutron stars collided, an electromagnetic wave and gravitational wave were simultaneously emitted. (We will ignore any possible delay between the two emissions.)

Assuming that electromagnetic waves consist of massless photons which move with the velocity of light c and that gravitational waves consist of gravitons of mass m and energy E, measuring the difference in arrival times of the two waves can be used to put a bound on mc^2/E.

At the time of the collision and in the reference frame of the center of mass of the neutron star collision, suppose the Earth is at a distance D' from the neutron stars and the Earth is moving away radially with constant velocity v_{Earth}.

1. **[10pt]** Compute mc^2/E in terms of (Δt,v_{Earth},D') where E is the energy of the graviton observed in the reference frame on Earth and $\Delta t = t - t_{\text{em}}$ is the difference in arrival times on Earth (t is the arrival time of the gravitational wave and t_{em} is the arrival time of the electromagnetic wave).

2. **[4pt]** Compute mc^2/E' in terms of (Δt,v_{Earth},D') where E' is the energy of the graviton emitted in the reference frame of the neutron star collision.

3. **[6pt]** Assuming that v_{Earth} is much less than c, compute the leading-order contribution in v_{Earth}/c to E'/E and compare with the leading-order contribution in v_{Earth}/c to $E'_{\text{em}}/E_{\text{em}}$ where E_{em} is the energy of the electromagnetic wave observed on Earth and E'_{em} is the energy of the electromagnetic wave emitted by the neutron star collision.

4. **[5pt]** In the 2017 neutron star collision, $D' = 1 \times 10^{24}$ meters and Δt was less than 2 seconds. Ignoring contributions from $\frac{v_{Earth}}{c}$, what is the bound on $\frac{mc^2}{E}$?

Some useful formulae: $E = m\gamma c^2$, $\gamma = 1/\sqrt{1 - (\frac{v}{c})^2}$, $c = 3 \times 10^8$ meters/second.

2 Inevitability of Bound-states

We will consider one and two dimensional Schrodinger problems $H\psi = E\psi$ where

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

in one dimension and

$$H = -\frac{\hbar^2}{2m}\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{d^2}{d\theta^2}\Big) + V(r,\theta)$$

in (polar coordinates in) two dimensions. We consider $V(x) \leq 0$ with $\lim_{x\to\pm\infty} V(x) = 0$ and $V(r,\theta) \leq 0$ with $\lim_{r\to\infty} V(r,\theta) = 0$ so that these potentials are attractive wells.

The goal of this problem is to show that for these attractive potentials there is a ground state with energy $E_0 < 0$. In other words, these potential wells in 1D and 2D always support at least one bound-state, no matter how shallow they might be. This is non-trivial: The analogue statement is not true in 3D for example.

Variational Method Recap

1. **[1pt]** Show that for any normalizable trial wave function ψ, in any dimension,

$$E_0 \leq \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \tag{1}$$

   Hint: You can consider ψ to be a linear combination of all eigenstates ψ_n including the ground-state ψ_0.

One dimension

The idea is to consider a trial wave function a trial gaussian wave function $\psi(x) = \mathcal{N}e^{-\alpha x^2}$ to show it produces a negative variational estimate of the energy. The following formula might be useful:

$$\int_{-\infty}^{\infty} e^{-Ax^2/2}dx = \sqrt{2\pi/A}\,.$$

2. **[1pt]** Show that

$$E_\alpha \equiv \frac{\int dx\,\psi^*(x)H\psi(x)}{\int dx\,\psi^*(x)\psi(x)} = \frac{\hbar^2\alpha}{2m} + \sqrt{\frac{2\alpha}{\pi}}\int e^{-2\alpha x^2}V(x)$$

3. **[2pt]** We want to α to be such that this quantity is minimized. What equation should α satisfy?

4. **[3pt]** Show that the minimum value of E_α, $E_\alpha^{\min}$, obeys

$$E_\alpha^{\min} = \sqrt{\frac{4\alpha}{2\pi}} \int dx (1 + 4\alpha x^2) e^{-2\alpha x^2} V(x)$$

5. **[3pt]** Show that all purely attractive wells support a bound-state in one dimensions.

Two dimensions

It will be enough to consider a radial trial wave functions $\psi(r, \theta) = \phi(r)$ with a real $\phi(r)$ which will depend on a variational parameter α. We consider normalizable wave functions and define

$$\mathcal{C}(\alpha) \equiv \langle \psi | \psi \rangle = 2\pi \int_0^\infty dr\, r\, \phi^2(r) < \infty\,.$$

Henceforth we set $\hbar = m = 1$.

6. **[3pt]** Show that

$$E_\alpha = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{1}{\mathcal{C}(\alpha)} \left[\pi \int_0^\infty dr\, r \left(\frac{d\phi(r)}{dr} \right)^2 + \int_0^{2\pi} d\theta \int_0^\infty dr\, r\, \phi(r)^2\, V(r, \theta) \right] \quad (2)$$

7. Consider a large class of trial wave functions of the form $\phi(r) = f(h(\alpha) r)$. for any function $h(\alpha) \geq 0$ and any function f.

   (a) **[1pt]** Show that this includes any wave function $\psi = \exp(-\alpha r^n)$ including the gaussian wave function.

   (b) **[2pt]** Show that for any such trial wave function $\phi(r) = f(h(\alpha) r)$ the first term in (2) (the kinetic term) is generically much larger than the second term in (2) (the potential terms) for very shallow potentials.

   (c) **[3pt]** Show that for shallow enough potentials such class of wave functions always yield $E_\alpha > 0$. What can we conclude about the existence (or not) of bound-states for shallow potentials?

8. Consider then

$$\phi(r) = e^{-(r + r_0)^\alpha} \quad (3)$$

with r_0 and α both positive. (r_0 ensures $\phi(r)$ always has a finite derivative at $r = 0$.) Note that this does *not* fit into the large class of potentials of the previous point. The variational energy (2) becomes

$$E_\alpha = \frac{1}{\mathcal{C}(\alpha)} \left[\pi \int_0^\infty dr\, r\, \alpha^2 (r + r_0)^{2\alpha - 2} e^{-2(r + r_0)^\alpha} + \int_0^{2\pi} d\theta \int_0^\infty dr\, r\, e^{-2(r + r_0)^\alpha} V(r, \theta) \right]$$

(a) **[1pt]** We focus first on the kinetic term. Show that

$$\int_0^\infty dr\, r\,(r+r_0)^{2\alpha-2} e^{-2(r+r_0)^\alpha} < \int_0^\infty dr\,(r+r_0)^{2\alpha-1} e^{-2(r+r_0)^\alpha} < \int_0^\infty dt\, e^{-2t^\alpha} t^{2\alpha-1}$$

(b) **[2pt]** Show that

$$E_\alpha \leq \frac{1}{\mathcal{C}(\alpha)} \left[\pi\alpha/4 + \int_0^{2\pi} d\theta \int_0^\infty dr\, r\, e^{-2(r+r_0)^\alpha} V(r,\theta) \right] \tag{4}$$

(c) **[3pt]** Show that all purely attractive wells support at least one bound-state in two dimensions.

3 Semi-Classical Quantization of the Dirac Monopole

The goal of this exercise is to semi-classically derive the quantization of the magnetic charge g.

Maxwell Equations with Magnetic Sources and the Magnetic Monopole

The Maxwell equations in the presence of both electric *and* magnetic sources read

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} , \quad \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} = \mu_0 \vec{j} , \tag{5}$$

$$\vec{\nabla} \cdot \vec{B} = \mu_0 \rho_m , \quad \vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}\vec{B} = \frac{\vec{j}_m}{\epsilon_0} . \tag{6}$$

1. [**4pt**] Show that with these new sources the modified Maxwell equations preserve "Maxwell duality": a symmetry under $\vec{E} \to \vec{B}$, $\vec{B} \to -\frac{1}{c^2}\vec{E}$. How do the sources transform under this symmetry?

2. [**5pt**] The Dirac monopole solution is the solution for a point-like magnetic source, the analog of the electron, satisfying

$$\vec{\nabla} \cdot \vec{B} = \mu_0 g \delta^3(\vec{x}). \tag{7}$$

   Write down the form of $\vec{B}$ for this solution.

Angular Momentum

The angular momentum stored in an electromagnetic field generalizes the momentum expression you might know as

$$\vec{L} = \epsilon_0 \int dV \, \vec{x} \times (\vec{E} \times \vec{B}) \tag{8}$$

We will compute this angular momentum for a system comprised of a point-like electric charge and a point-like magnetic charge separated by $\vec{a}$ as depicted in figure 1.

4. [**4pt**] Show that

$$\vec{L} = \epsilon_0 \int dV \, \vec{x} \times (\vec{\nabla}\phi \times \vec{\nabla}\bar{\phi}) \tag{9}$$

   where $\phi = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{x}-\vec{a}/2|}$ is the electric potential obeying

$$\nabla^2 \phi = -\frac{q}{\epsilon_0}\delta^3(\vec{x} - \vec{a}/2) . \tag{10}$$

   What is $\bar{\phi}$ and what equation does it satisfy?

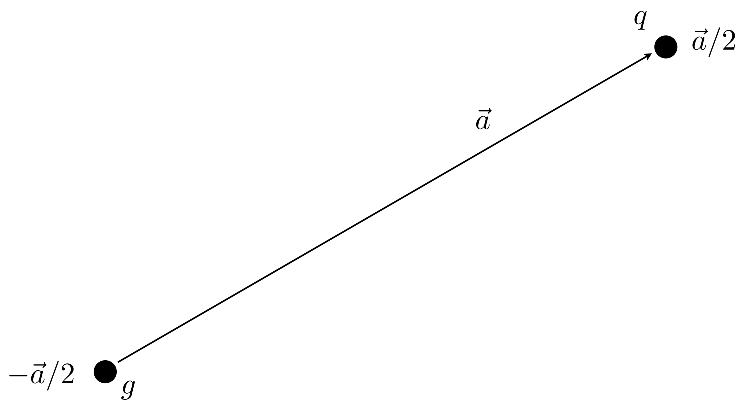


Figure 1: Thomson's dipole: An electric point particle at $+\vec{a}/2$ with electric charge q and a magnetic monopole at $-\vec{a}/2$ and magnetic charge g.

5. **[3pt]** Introducing the vector

$$\vec{C} \equiv \frac{\epsilon_0}{2}(\phi\vec{\nabla}\bar{\phi} - \bar{\phi}\vec{\nabla}\phi) \tag{11}$$

with the nice property $\vec{\nabla} \times \vec{C} = \epsilon_0 \vec{\nabla}\phi \times \vec{\nabla}\bar{\phi}$ and using the identity

$$\vec{x} \times (\vec{\nabla} \times \vec{C}) = \vec{\nabla}(\vec{C} \cdot \vec{x}) - \vec{\nabla} \cdot (\vec{x}\vec{C}) + 2\vec{\nabla} \cdot (\vec{C}\vec{x}) - 2\vec{x}(\vec{\nabla} \cdot \vec{C}) \tag{12}$$

show that

$$\vec{L} = -2 \int dV \, \vec{x}(\vec{\nabla} \cdot \vec{C}) \, . \tag{13}$$

6. **[5pt]** Show that

$$\vec{L} = \frac{\mu_0}{4\pi} gq \, \hat{a} \tag{14}$$

where $\hat{a}$ is the unit vector in the direction $\vec{a}$ connecting the magnetic and electric charge. We conclude that the magnitude of the angular momentum is $\frac{\mu_0}{4\pi} gq$.

7. **[4pt]** Argue that quantum mechanically we should have

$$qg = Nh/\mu_0 \tag{15}$$

where $N = 1, 2, \ldots$. Here h is Plank's constant (not the reduced one).

This is Dirac's quantization condition for the monopole. It is usually derived quantum mechanically by imposing proper transformations under gauge transformations of electron wave functions in the presence of a monopole. Here we argued for it semi-classically.

4 Neutron Star

Consider a noninteracting gas of indistinguishable quantum particles of mass M at temperature T contained in a large volume V. The average occupation number $n(\boldsymbol{k})$ of a single-particle state of energy $E(\boldsymbol{k})$ can be written as

$$n(\boldsymbol{k}) = \frac{g}{e^{[E(\boldsymbol{k})-\mu]/k_B T} \pm 1}, \tag{16}$$

where g is the degeneracy of the single-particle energy level, k_B the Boltzmann constant, and μ the chemical potential. The $\pm$ signs in the denominator in Eq. (16) distinguish Fermi and Bose gases.

1. [**2.5pt**] Show that for a Bose gas with single-particle energies $E(\boldsymbol{k}) = \hbar^2 \boldsymbol{k}^2/2M$ the chemical potential μ is necessarily negative.

2. [**2.5pt**] Given that μ gives the change in the internal energy of the gas when one more particle is added while holding the volume and the entropy constant, what is the physical meaning of a negative μ?

   Hint: Recall the thermodynamic identity $dU = T\,dS - p\,dV + \mu\,dN$ (valid for infinitesimal, reversible processes).

3. [**2.5pt**] Bose-Einstein condensation is the phenomenon that at sufficiently low temperatures, all bosons occupy the lowest energy state; the bosons form a "condensate". When all bosons condense in the zero-energy state, the condensate has $\mu = 0$. Use entropic arguments to explain why $\mu = 0$ in such a situation.

4. [**2.5pt**] Consider nonrelativistic neutrons with single-particle energies given by $E(\boldsymbol{k}) = \hbar^2\boldsymbol{k}^2/2M$, where M is the neutron mass. Show that the chemical potential of the neutron star is equal to the Fermi energy, $\mu = E_F$ where $E_F = \hbar^2 k_F^2/2M$ and $k_F = (3\pi^2\rho)^{1/3}$, with $\hbar k_F$ being the Fermi momentum.

5. [**7.5pt**] Suppose the single-neutron energies are given by the relativistic expression $E(\boldsymbol{k}) = \sqrt{\hbar^2\boldsymbol{k}^2c^2 + M^2c^4}$. Show that the neutron star pressure of the gas reads:

$$P = \frac{k_F^3}{3\pi^2}(\hbar k_F c)^2 \int_0^1 dx\, \frac{x^4}{(\hbar^2c^2k_F^2x^2 + M^2c^4)^{1/2}} = \begin{cases} \dfrac{2}{5}\,\rho\, E_F, & \text{nonrelativistic neutrons}, \\[2ex] \dfrac{M^4c^5}{12\pi^2\hbar^3}\left(\dfrac{\hbar k_F}{Mc}\right)^4, & \text{ultrarelativistic neutrons}. \end{cases} \tag{17}$$

   Hint: recall that the pressure is $P = -\partial E/\partial V$, where E is the total energy.

One can find an analytical expression for the equivalent of the Chandrasekhar limit for such a hypothetical neutron star in Newtonian gravitation. This is the maximum mass for a pure neutron star described by the nointeracting Fermi gas. Above this limit, the gravitational pressure would overcome the Fermi gas pressure and the star would collapse. For ultrarelativistic neutrons, this maximum mass can be written only in terms of the fundamental constants $\hbar$, c and G, and the neutron mass M.

6. **[7.5pt]** First, use Newton's law of gravitation for the differential of the force per unit area $dF/A = dP$ to show that for a spherical homogeneous star with constant mass density $n_* = M_*/V$ we have

$$\frac{dP(r)}{dr} = -\frac{GM_*(r)}{r^2}\, n_* \tag{18}$$

where $M_*(r)$ is the mass contained in a spherical volume of radius r. Next, integrate this equation to show that the maximum mass of the star for ultrarelativistic neutrons is given by

$$M_*^{\text{ult.rel.}} = \frac{3}{8}\left[\frac{2\pi}{M^4}\left(\frac{\hbar c}{G}\right)^3\right]^{1/2} . \tag{19}$$