Emergent Phenomena in Quantum Matter $\boxed{0}$

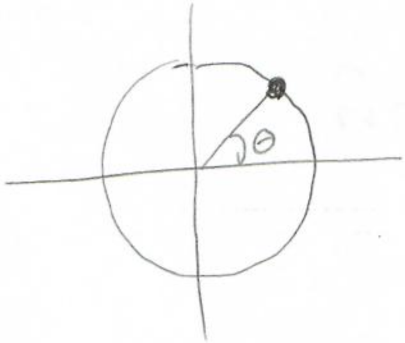
- Cooperative behaviors of many degrees of freedom that arise only in the $V \to \infty$ limit

- plan

1. quantum rotor model
2. path integral representation of ground state
3. insulating phase
4. Spontaneous symmetry breaking
5. topological order.

(*) 6. Quantum criticality (if time permits)

1. rotor model



$0 \le \theta < 2\pi$

$$[\hat{p}_\theta, \hat{\theta}] = -i\hbar \quad ([p, x] = -i\hbar)$$

$\hat{\theta}$ ← angle

$\hat{p}_\theta$ ← angular momentum

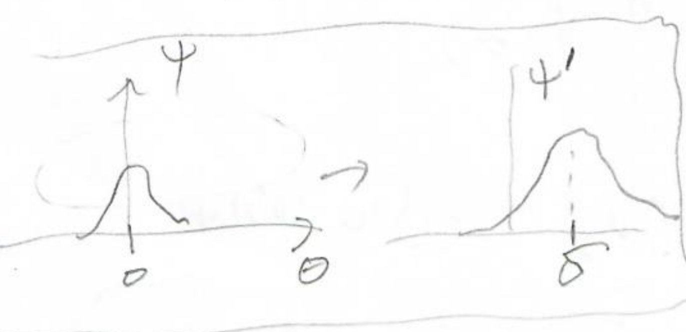
In the angle basis, $\hat{p}_\theta = -i\hbar \frac{\partial}{\partial \theta}$ $\quad \hat{n} \equiv \frac{1}{\hbar}\hat{p}_\theta = -i\frac{\partial}{\partial \theta}$

- eigenstate of $\hat{\theta}$: $\psi_{\theta_1}(\theta) = \delta(\theta - \theta_1)$.
- eigenstates of $\hat{n}$; $\psi_n(\theta) = e^{in\theta}$.

In order for the wavefunction to be periodic $n \in \mathbb{Z}$.

- $\{\psi_{\theta_1}(\theta)\}, \{\psi_n(\theta)\}$ form complete basis.
- $\hat{n}$ is the generator of angular translation ;

$$\psi(\theta) \to \psi(\theta - \delta) = \underbrace{\sum_{l=0}^{\infty} \frac{1}{l!}(-\delta)^l \left(\frac{d}{d\theta}\right)^l}\psi(\theta)$$



$$= e^{-\delta \frac{d}{d\theta}} \psi(\theta)$$

$$= e^{-i\delta \hat{n}} \psi(\theta).$$

$\therefore$ $e^{-i\delta\hat{n}}\psi(\theta)$ represents the state where the angle is rotated by δ (U(1) transformation)

$\boxed{\text{U(1) group}}$ $\quad \{ e^{-i\delta\hat{n}} \mid 0 \le \delta < 2\pi \}$ $\quad \boxed{1}$-①

$(\because)$ $e^{-i0\cdot\hat{n}} = I$

$$e^{-i\delta_1\hat{n}} e^{-i\delta_2\hat{n}} = e^{-i(\delta_1+\delta_2)\hat{n}}$$

$$e^{-i\delta_1\hat{n}} e^{i(\delta_1)\hat{n}} = I$$

$$e^{-i2\pi\hat{n}} = I \qquad \Rightarrow \text{U(1) group.}$$

- A Hamiltonian H is said to have the U(1) symmetry if $[H,\hat{n}]=0$

Implication ① $\int d\theta\, \psi^*(\theta)\hat{H}\psi(\theta) = \int d\theta\, \psi'^*(\theta) H \psi'(\theta)$

$\left(\psi'(\theta) = e^{-i\delta n}\psi(\theta) \right)$

$$\int d\theta \left[e^{-i\delta n}\psi(\theta)\right]^* H e^{-i\delta n}\psi(\theta)$$

$$= \int d\theta\, \psi^*(\theta)\, e^{i\delta n} H e^{-i\delta n}\psi(\theta)$$

$$= \int d\theta\, \psi^* \left(H - i\delta[H,n]\right)\psi = \int d\theta\, \psi^* H \psi$$

Implication

② $\int d\theta\, \psi^*(\theta,t)\,\hat{n}\,\psi^*(\theta,t) = \int d\theta\, \psi^*(\theta,0)\,\hat{n}\,\psi(\theta,0)$

* A state is said to have the U(1) symmetry if

$\to$ $e^{-i\delta\hat{n}}\psi = \psi$ or $\hat{n}\psi = 0$

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Ex) $e^{\pm i\theta}$ raises / lowers the angular momentum

$$[n, e^{\pm i\theta}] = [-i\frac{\partial}{\partial\theta}, e^{\pm i\theta}] = \pm e^{\pm i\theta}$$

$$\therefore \hat{n} e^{\pm i\theta} \psi_m(\theta) = (e^{\pm i\theta}\hat{n} \pm e^{\pm i\theta}) \psi_m(\theta)$$

$$= (m \pm 1) e^{\pm i\theta} \psi_m(\theta)$$

- Hamiltonian that preserves the angular momentum (charge) can not depend on θ.

(i) $V(\theta) = \sum_n V_n e^{in\theta}$.

terms with $n \neq 0$ will create / remove the angular momentum

$$\therefore H = f(\hat{n}) = f_0 + f_1 \hat{n} + f_2 \hat{n}^2 + \cdots$$

$\uparrow$ breaks the time-reversal

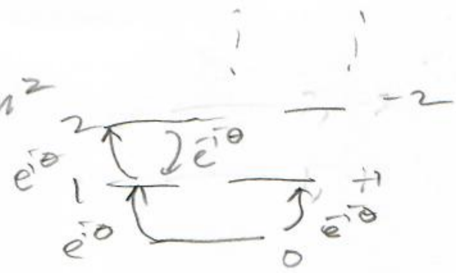
$\because$ $n = \pm 1$ have different energy.

the simplest H that respects angular momentum conservation & T-reversal :

$$\boxed{H = \frac{U}{4}\hat{n}^2}$$

eigenstate

$\psi_m(\theta) = \frac{1}{\sqrt{2\pi}} e^{im\theta}$, $E_m = \frac{U}{4} m^2$



· Let's consider many-body system with net angular momentum conservation & T-reversal.

$$\boxed{H = \frac{U}{4}\sum_i \hat{n}_i^2 - \sum_{ij} t_{ij}\, e^{i\hat{\theta}_i} e^{-i\hat{\theta}_j} + \cdots}$$

kinetic energy (favours $\hat{n} = 0$)

hopping term (creates fluctuation in $\hat{n}$)

· the simplest local Hamiltonian:

$$\boxed{H = \frac{U}{4}\sum_i \hat{n}_i^2 - t\sum_{ij} e^{i(\hat{\theta}_i - \hat{\theta}_j)}}$$

(ordered nearest neighbor pairs in a lattice.)

L: linear system size, $V = L^d$; total # of sites, d: dimension

* Goal: to understand collective behaviours of this model that only emerge in the $V \to \infty$ limit at $T = 0$.

In general, we can not obtain the exact ground state.

One way to obtain ground states is to start with a 'seed state' and 'cool' it.

$$\Psi_{G.S}(\theta) = \underbrace{\lim_{T\to\infty} e^{-T\hat{H}}}_{\text{project out excited states}} \Psi_s(\theta)$$

$(\theta_1 \cdots \theta_V)$

true ground state.

seed state

$$\left[(\because) \quad e^{-TH}\left(\sum_m C_m \Psi_m(\theta)\right) = \sum_m C_m e^{-TE_m}\Psi_m \xrightarrow[\text{large } T]{} \sum_{m' \in G.S} C_{m'} e^{-TE_0}\Psi_{m'} \right]$$

Let $H_U = \sum_i \frac{U}{4}\hat{n}_i^2$, $H_t = -t\sum_{ij} e^{i(\theta_i-\theta_j)}$

$\boxed{4}$-①

Proof of $\boxed{5}$-① :

$$= e^{-\varepsilon \hat{H}_U(\hat{n})} e^{-\varepsilon \hat{H}_t(\hat{\theta})} \psi_s(\theta)$$

$$= e^{-\varepsilon \hat{H}_U(\hat{n})} e^{-\varepsilon H(\theta)} \psi_s(\theta)$$

$$= e^{-\varepsilon \hat{H}(\hat{n})} \sum_{n^{(1)}} e^{i n^{(1)}\theta} \cdot f_{n^{(1)}} \quad \left[f_{n^{(1)}} = \int \frac{d\theta^{(1)}}{2\pi} e^{-i n^{(1)}\theta^{(1)}} e^{-\varepsilon H_t(\theta^{(1)})} \psi_s(\theta^{(1)}) \right]$$

$$= \sum_{n^{(1)}} \int \frac{d\theta^{(1)}}{2\pi} e^{i n^{(1)}(\theta-\theta^{(1)}) - \varepsilon\left[H_U(n^{(1)}) + H_t(\theta^{(1)})\right]} \psi_s(\theta^{(1)})$$

$\boxed{4}$-①-$\circledast$

2. Path Integral

$\hat{H}$ includes terms that do not commute.
so $\hat{H}_U$, $\hat{H}_t$ can not be simultaneously minimized.

→ quantum fluctuations.

$$\Psi_{G,S}(\theta) = \lim_{T\to\infty} e^{-\varepsilon\hat{H}} e^{-\varepsilon\hat{H}} \cdots\cdots e^{-\varepsilon\hat{H}} \Psi_S(\theta)$$

We first consider one infinitesimal step of the imaginary time evolution:

$$e^{-\varepsilon\hat{H}}\Psi_S(\theta) = \left[e^{-\varepsilon H_U(\hat{n})} e^{-\varepsilon H_t(\hat{\theta})} + O(\varepsilon^2)\right]\Psi_S(\theta)$$

From the identity derived in $\boxed{4}$-①,
we obtain

$$\sum_{n_i^{(1)}\cdots n_i^{(V)}} \qquad \int \frac{d\theta_1^{(1)}}{2\pi}\cdots\frac{d\theta_V^{(1)}}{2\pi}$$

site index i summed over

$$e^{-\varepsilon\hat{H}}\Psi_S(\theta) = \sum_{n_i^{(1)}} \int \frac{d\theta_i^{(1)}}{2\pi} e^{i n_i^{(1)}(\theta_i - \theta_i^{(1)}) - \varepsilon H(n^{(1)}, \theta^{(1)})} \Psi_S(\theta^{(1)}) \quad \boxed{5}\text{-①}$$

$$\therefore\ e^{-\varepsilon\hat{H}} e^{-\varepsilon\hat{H}} \Psi(\theta)$$

$$= \sum_{n_i^{(2)} n_i^{(1)}} \int \frac{d\theta_i^{(2)} d\theta_i^{(1)}}{(2\pi)^{2V}} e^{i n_i^{(2)}(\theta_i - \theta_i^{(2)}) - \varepsilon H(n^{(2)}, \theta^{(2)})}$$
$$\times\ e^{i n_i^{(1)}(\theta_i^{(2)} - \theta_i^{(1)}) - \varepsilon H(n^{(1)}, \theta^{(1)})} \Psi_S(\theta^{(1)})$$

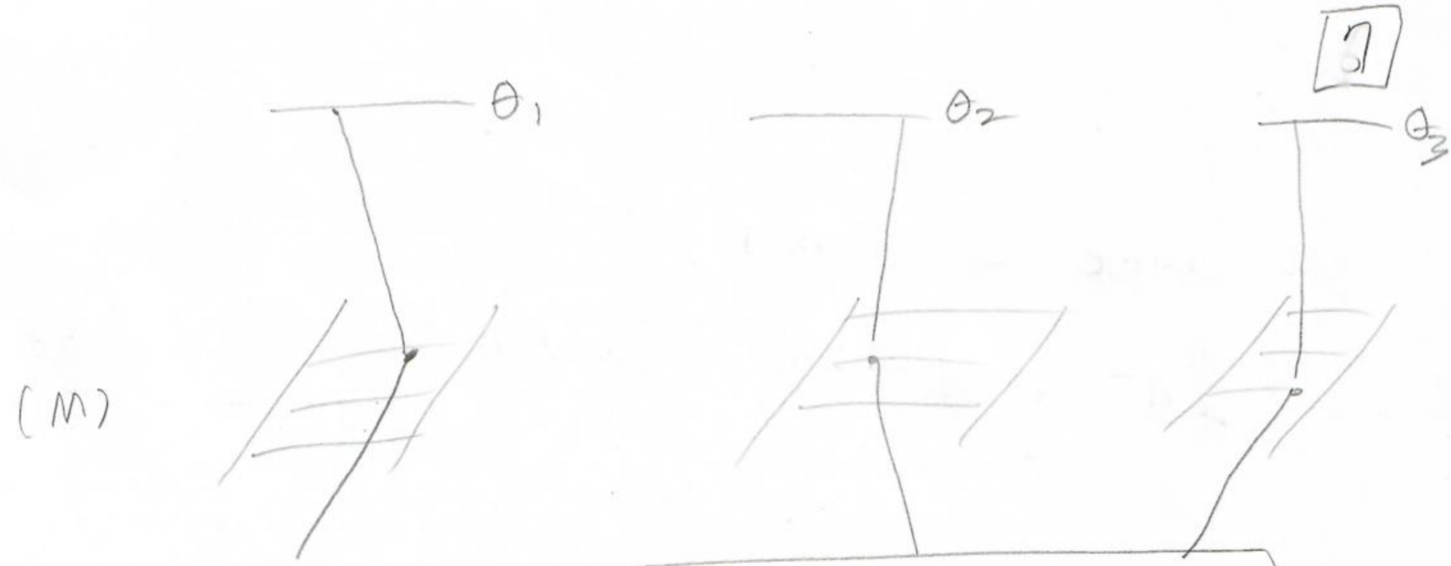
$$\Psi_{G_f}(\theta) = \lim_{T\to\infty} \sum_{n_i^{(1)}\cdots n_i^{(M)}} \int \frac{d\theta_i^{(1)}\cdots d\theta_i^{(M)}}{(2\pi)^{VM}} e^{i\sum_{l=1}^{M} n_i^{(l)}(\theta_i^{(l+1)}-\theta_i^{(l)})} \times$$

$$e^{-\varepsilon \sum_{l=1}^{M} H(n^{(l)},\theta^{(l)})} \quad \Psi_s(\theta^{(1)}),$$

with $\theta_i = \theta_i^{(M+1)}$

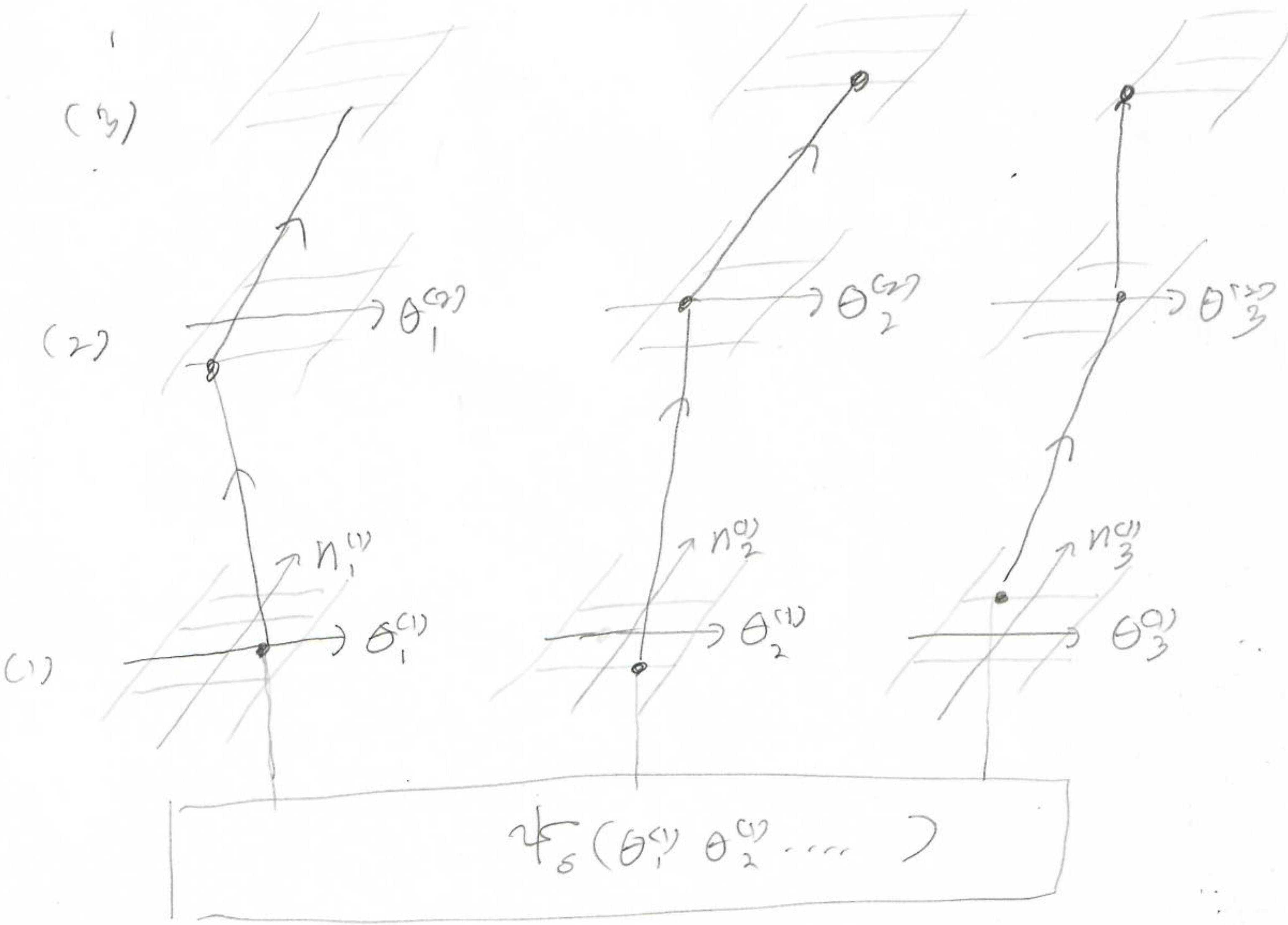
$$\Psi_{G_f^s}(\theta(T)) \equiv \lim_{T\to\infty} \sum_{n} \int \mathcal{D}\theta \; e^{i\int_0^T d\tau\, n_i(\tau)\,\partial_\tau \theta_i - \int_0^T d\tau\, H(n,\theta)} \; \Psi_s(\theta(0))$$

$\sum_n \int \mathcal{D}\theta$ sums over all paths that $\{n_i(\tau), \theta_i(\tau)\}$ can take along "time" 0 to T.



each path is weighted with e^{-S}

$$S = -i\int_0^T d\tau\, n_i \partial_\tau \theta_i + \int d\tau\, H(\theta(\tau), n(\tau))$$



3. Strong coupling limit ($U \gg t$)

- In the ground state, the fluctuations of n (θ) are small (large).
- It is more convenient to integrate out θ to understand the physics

$$\Psi_{G,S}(\theta) = \lim_{T\to\infty} \sum_{n^{(\ell)}} \int D\theta \; e^{\sum_{\ell=1}^{M}\left[i n_i^{(\ell)}(\theta_i^{(\ell+1)} - \theta_i^{(\ell)}) + \varepsilon t \sum_{ij} e^{i(\theta_i^{(\ell)} - \theta_j^{(\ell)})} - \frac{U}{4}\varepsilon (n_i^{(\ell)})^2 \right]} \Psi_S(\theta^{(1)})$$

$$= \lim_{T\to\infty} \sum_{n^{(\ell)}} \int D\theta \; e^{i n_i^{(M)}\theta_i - i\sum_{\ell=2}^{M}\theta_i^{(\ell)}(n_i^{(\ell)} - n_i^{(\ell-1)}) - i n_i^{(1)}\theta_i^{(1)}} \; e^{-\frac{U}{4}\varepsilon\sum_\ell (n_i^{(\ell)})^2}$$

$$\prod_{\ell}\prod_{ij}\left(1 + \varepsilon t\, e^{i(\theta_i^{(\ell)} - \theta_j^{(\ell)})}\right) \qquad \underbrace{\Psi_S(\theta^{(1)})}_{\displaystyle \sum_{\{n_i^{(0)}\}} \Psi_{S,\{n_i^{(0)}\}} e^{i\sum_i n_i^{(0)}\theta_i^{(0)}}}$$

$$= \lim_{T\to\infty} \sum_{n^{(\ell)}} e^{i n_i^{(M)}\theta_i} \; e^{-\frac{U}{4}\int d\tau\, (n_i(\tau))^2} \; \Psi_{S,\{n^{(0)}\}} \times$$

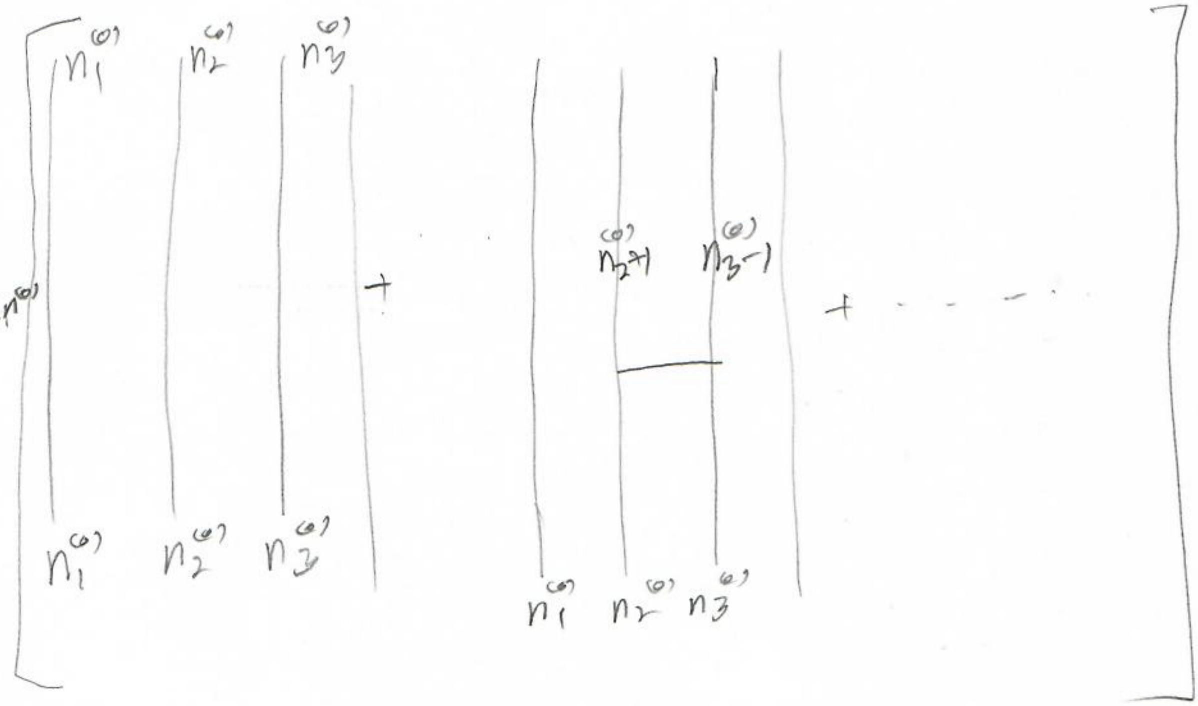
$$\int D\theta \; e^{-i\sum_{\ell=1}^{M}\theta_i^{(\ell)}(n_i^{(\ell)} - n_i^{(\ell-1)})}$$

$$\left[1 + t\sum_{(i_1 j_1)}\int d\tau_1\, e^{i(\theta_{i_1}(\tau_1) - \theta_{j_1}(\tau_1))} + \frac{t^2}{2}\sum_{\substack{(i_1 j_1)\\(i_2 j_2)}}\int d\tau_1 d\tau_2\, e^{i(\theta_{i_1}(\tau_1) - \theta_{j_1}(\tau_1)) + i(\theta_{i_2}(\tau_2) - \theta_{j_2}(\tau_2))} + \cdots \right]$$

$$= \lim_{T\to\infty} \sum_{n^{(0)}} e^{i n_i\theta_i} \left[e^{-\frac{U}{4}\int d\tau\, n_i^2} + t\int_0^T d\tau_1 \sum_{i_1 j_1} e^{i(\theta_{i_1}(\tau_1) - \theta_{j_1}(\tau_1))} \; e^{-\frac{U}{4}\sum_i\int_0^{\tau_1} d\tau\, n_i^2 - \frac{U}{4}\sum_i\int_{\tau_1}^{T} d\tau\,(n_i + \Delta n_i)^2} + \cdots \right]$$

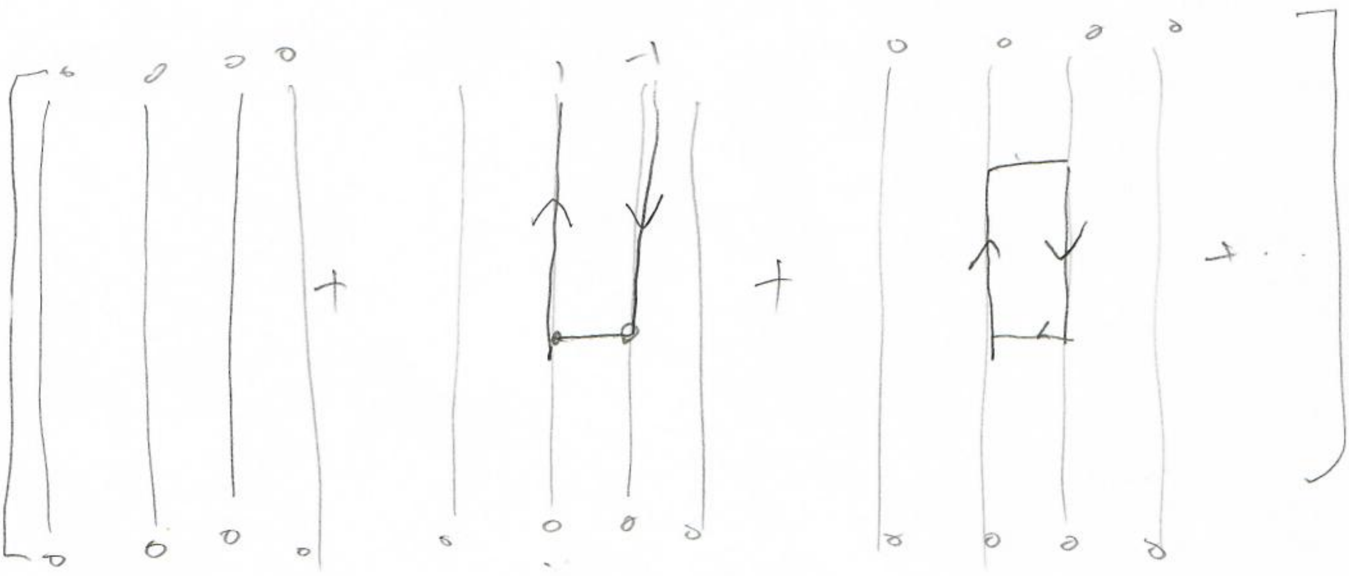
$$\left(\Delta n_i = \begin{matrix} 1 & i = i_1 \\ -1 & = j_1 \end{matrix} \right)$$

$\sum_{n_i^{(0)}} \Psi_{S,n^{(0)}}$

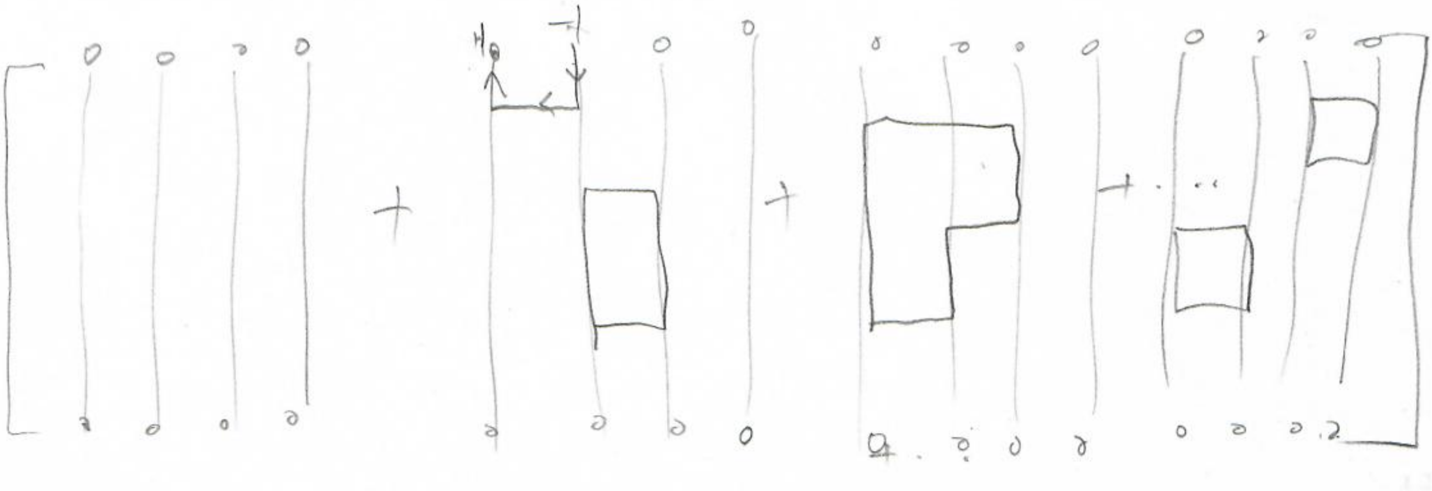


In the large U limit, only the sector with $\sum n_i = 0$ survive.

$\approx \Psi_{S,0}$



$\approx \Psi_{S,0}$



In the large U/t limit, the ground state is unique,

$$\Psi_G = \prod_i \psi_0(\theta_i) + \sum_{ij} C_{ij} \underbrace{\left[\psi_1(\theta_i)\,\psi_{-1}(\theta_j)\right]}_{\text{(local charge fluctuations)}} \left[\prod_{k\neq i,j} \psi_0(\theta_k)\right] + \dots$$

$$C_{ij} \sim t \int_0^T d\tau\, e^{-\frac{U}{2}\cdot\tau} \sim \left(\frac{t}{U}\right) \quad \text{for } \langle i,j\rangle : \text{nearest neighbors.}$$

In general,

$$C_{ij} \sim \left(\frac{t}{U}\right)^{|i-j|}$$

4. Weak coupling limit ($t \gg U$)

In this case, we expect large fluctuations in n_i, and we integrate out n_i to obtain a path integral for the angle variables.

$$\sum_{n=-\infty}^{\infty} e^{in(\Delta\theta) - \frac{\varepsilon U}{4} n^2} = \theta_3\left(\frac{\Delta\theta}{2}, e^{-\frac{\varepsilon U}{4}}\right) \approx \sqrt{\frac{4\pi}{\varepsilon U}}\, e^{-\frac{(\Delta\theta)^2}{\varepsilon U}}.$$

(Ex) Without loss of generality, $-\pi < \Delta\theta \leq \pi$

$$\sum_{n=-\infty}^{\infty} e^{in\Delta\theta - \varepsilon\frac{U}{4}n^2}$$

$$= \int dn \underbrace{\sum_m e^{2\pi i m \cdot n}}_{\sum_k \delta(n-k)} e^{+in\Delta\theta - \varepsilon\frac{U}{4}n^2}$$

$$= \sum_m \int dn\, e^{-\frac{\varepsilon U}{4}\left(n - \frac{2i\Delta'\theta}{U\varepsilon}\right)^2 - \frac{(\Delta'\theta)^2}{U\varepsilon}} \qquad \Delta'\theta = \Delta\theta + 2\pi m$$

$$= \sqrt{\frac{4\pi}{\varepsilon U}} \sum_m e^{-\frac{(\Delta\theta + 2\pi m)^2}{U\varepsilon}} \approx \sqrt{\frac{4\pi}{\varepsilon U}}\, e^{-\frac{(\Delta\theta)^2}{U\varepsilon}} \quad \text{for } |\Delta\theta| < \pi$$

In the small ε limit, the contribution from $|\Delta\theta| \gg \sqrt{\varepsilon}$ will vanish.

$$\boxed{\Psi_{G,S} = \lim_{T\to\infty} \int d\theta^{(1)} \ldots d\theta^{(M)}\, e^{-\varepsilon \sum_{\ell=1}^{M}\left[\frac{1}{U}\left(\frac{\theta_i^{(\ell+1)}-\theta_i^{(\ell)}}{\varepsilon}\right)^2 + t\sum_{ij} e^{i(\theta_i^{(\ell)}-\theta_j^{(\ell)})}\right]} \Psi_S(\theta^{(1)})}$$

$$\Psi_{G,S}(\theta) = \lim_{T\to\infty} \int D\theta\, e^{-\int_0^T d\tau \left[\sum_i \frac{1}{U}\left(\frac{\partial\theta_i}{\partial\tau}\right)^2 - t\sum_{ij} e^{i(\theta_i-\theta_j)}\right]} \Psi_S(\theta(0))$$

$$\tau \to \frac{1}{\sqrt{Ut}}\tau \qquad \left(\frac{1}{g^2} = \sqrt{\frac{t}{U}}\right)$$

$$= \lim_{T\to\infty} \int_{\theta(T)=\theta} D\theta\, e^{-\frac{1}{g^2}\int_0^T d\tau \left[\sum_i \left(\frac{\partial\theta_i}{\partial\tau}\right)^2 - t\sum_{ij} e^{i(\theta_i-\theta_j)}\right]} \Psi_S(\theta(0))$$

by shifting $\tau \to \tau - T$

$$= \lim_{T\to\infty} \int_{\theta(0)=\theta} D\theta\, e^{-\frac{1}{g^2}\int_{-T}^0 d\tau \left[\sum_i \left(\frac{\partial\theta_i}{\partial\tau}\right)^2 - t\sum_{ij} e^{i(\theta_i-\theta_j)}\right]} \Psi_S(\theta(0)).$$

Let's choose the seed state to be

$$\Psi_S(\theta) = \prod_j \delta(\theta_j - \varphi)$$

and compute the expectation value of the angle at a site

(EX)

$$\langle e^{i\theta_0}\rangle = \lim_{T\to\infty} \int D\theta\, e^{-\frac{1}{g^2}\left[\int_{-T}^0 d\tau\, L + \int_0^T d\tau\, L\right]} e^{i\theta_0(0)} \Bigg|_{\theta(T)=\theta(-T)=\varphi}$$

$$= \lim_{T\to\infty} \int D\theta\, e^{-\frac{1}{g^2}\int_{-T}^T d\tau \left[\sum_i \left(\frac{\partial\theta_i}{\partial\tau}\right)^2 - t\sum_{ij} e^{i(\theta_i-\theta_j)}\right] + i\theta_0(0)} \Bigg|_{\theta(T)=\theta(-T)=\varphi}$$

For $g^2 \ll 1$, the dominant contribution to the path integral comes from the saddle point path $\theta_j(\tau) = \varphi$

$$\theta = \varphi + \underset{\text{small fluctuation.}}{\theta'}$$

$$-t\sum_{ij} e^{i(\theta_i - \theta_j)} = -2t\sum_{\langle ij\rangle} \cos(\theta_i - \theta_j)$$

$\langle ij\rangle$: unordered pairs of nearest neighbour sites

$$= t\sum_{\langle ij\rangle} (\theta_i - \theta_j)^2 + \text{const.}$$

$$\langle e^{i\theta_0}\rangle = e^{i\varphi} \lim_{T\to\infty} \int D\theta'\, e^{-\frac{1}{g^2}\int_{-T}^{T} d\tau \left[\sum_i \left(\frac{\partial\theta_i'}{\partial\tau}\right)^2 + \sum_i\sum_\mu (\theta_{i+\mu}' - \theta_i')^2\right] + i\theta_0'(0)}$$

($\mu = 1, \cdots, d$: direction in space)

$$\theta_i' = \frac{1}{2T}\frac{1}{V}\sum_{\omega_n}\sum_{\vec{k}} \sin\omega_n(\tau+T)\, e^{i\vec{k}\cdot\vec{r}_i}\, \theta'_{\omega\vec{k}}$$

$$\left(\begin{array}{ll} 2T\omega_n = \pi n \quad n = 1, 2, \cdots & \omega = \dfrac{\pi n}{2T} \\ \vec{k} = \dfrac{2\pi}{L}(b_1, b_2, \cdots), \quad -\dfrac{L}{2} \le b_i < \dfrac{L}{2}. & \end{array}\right)$$

$$\theta_0'(0) = \frac{1}{2TV}\sum_{\omega_n}\sum_{\vec{k}} \sin\frac{\pi n}{2}\, \theta'_{\omega_n\vec{k}}$$

$$\sum_i \int_{-T}^{T} d\tau \left[\left(\frac{\partial\theta_i'}{\partial\tau}\right)^2 + \sum_\mu (\theta_{i+\mu}' - \theta_i')^2\right]$$

$$= \sum_i \int_{-T}^{T} d\tau \left(\frac{1}{2TV}\right)^2 \sum_{\omega\vec{k}}\sum_{\omega'\vec{k}'} \left[\begin{array}{l} \omega\omega'\cos\omega(\tau+T)\cos\omega'(\tau+T) \\ + \sum_\mu (e^{ik_\mu} - 1)(e^{ik_\mu'} - 1)\sin\omega(\tau+T)\sin\omega'(\tau+T) \end{array}\right]$$

$$\times\, e^{i\vec{k}\cdot\vec{r}_i + i\vec{k}'\cdot\vec{r}_i}\, \theta'_{\omega\vec{k}}\,\theta'_{\omega'\vec{k}'}$$

From $\sum_i e^{i(\vec{k}+\vec{k}')\cdot\vec{r}_i} = V\delta_{\vec{k},-\vec{k}'}$

$$\int_{-T}^{T} dz \begin{bmatrix} \cos\omega(z+T)\cos\omega'_n(z+T) \\ \sin\omega(z+T)\sin\omega'_n(z+T)\end{bmatrix} = T\,\delta_{\omega\omega'} ,$$

$$\therefore \sum_i \int_{-T}^{T} dz \left[\left(\frac{\partial\theta'_i}{\partial z}\right)^2 + \sum_\mu (\theta'_{i+\mu}-\theta'_i)^2\right] = \frac{1}{4TV}\sum_{\substack{\omega_m \vec{k}}} \Big[\omega_m^2 + \overbrace{\sum_\mu (2-2\cos k_\mu)}^{\epsilon_k}\Big] |\theta'_{\omega_m \vec{k}}|^2$$

$$\therefore \langle e^{i\theta_0}\rangle = e^{i\varphi}\lim_{T\to\infty}\int D\theta'_{\omega k}\, e^{-\frac{1}{2TV}\sum_{\omega_m,k}\underbrace{\left[\frac{\omega_m^2+\epsilon_k}{2g^2}|\theta'_{\omega_m k}|^2 - i\,\theta'_{\omega_m,k}\sin\frac{\pi m}{2}\right]}}$$

$\left(\Delta \equiv \frac{\omega_m^2+\epsilon_k}{g^2}\right)$

(Ex)

$$\sum_{m=0(\text{odd})}\left[\frac12\Delta_{\omega k}\left(\theta'^*_{\omega k} - \frac{i(-1)^{\frac{m-1}{2}}}{\Delta_{\omega k}}\right)\left(\theta'_{\omega k}-\frac{i(-1)^{\frac{m-1}{2}}}{\Delta_{\omega_m k}}\right) + \frac{1}{2\Delta_m}\right]$$

$$= e^{i\varphi}\lim_{T\to\infty} e^{-\underbrace{\frac{1}{4TV}\sum_{\substack{k\,\omega_n \\ m=\text{odd}}}\frac{g^2}{\omega_m^2+\epsilon_k}}} \quad (*) \qquad \Delta\omega = \frac{2\pi}{2T}$$

For finite V,

$$T\to\infty: \quad (*) = e^{-\frac{1}{4V}\sum_k\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{g^2}{\omega^2+\epsilon_k}} = e^{-\frac{g^2}{8V}\sum_k\frac{1}{\sqrt{\epsilon_k}}} \to 0 \quad k=0$$

(Alternatively,

$$\langle e^{i\theta_0}\rangle \le e^{i\varphi - \frac{g^2}{8V}\frac{1}{T}\left[\frac{1}{(\frac{\pi}{2T})^2}\right]} = e^{i\varphi-\frac{g^2}{2\pi^2}\frac{T}{V}} \xrightarrow[\lim T\to\infty]{} 0 \;)$$

$\therefore$ For any finite system, the ground state does NOT keep the memory of the seed state $\Rightarrow$ unique ground state for finite V.

If we take $V \to \infty$ limit first before $T \to \infty$ limit,

$$\therefore \langle e^{i\theta_0} \rangle = e^{i\theta - \frac{g^2}{8}\int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{\epsilon_{\vec{k}}}}}.$$

the momentum integration is UV finite.

However, it may have IR divergence near $\vec{k} = 0$.

expanding ϵ_k near $k = 0$

$$\epsilon_k = |\vec{k}|^2 + O(k^4)$$

$$\therefore \int \frac{d^d k}{(2\pi)^d} \frac{1}{k} \sim \frac{S_{d-1}}{(2\pi)^d} \int_0^{\pi} dk \, k^{d-2} \sim \begin{cases} \frac{S_{d-1}}{(2\pi)^d (d-1)} \pi^{d-1} & \text{for } d > 1 \\ \log \pi/0 \to \infty & d = 1 \\ 0^{-1} \to \infty & d = 0 \end{cases}$$

(S_{d-1}: volume of $(d-1)$-sphere.)

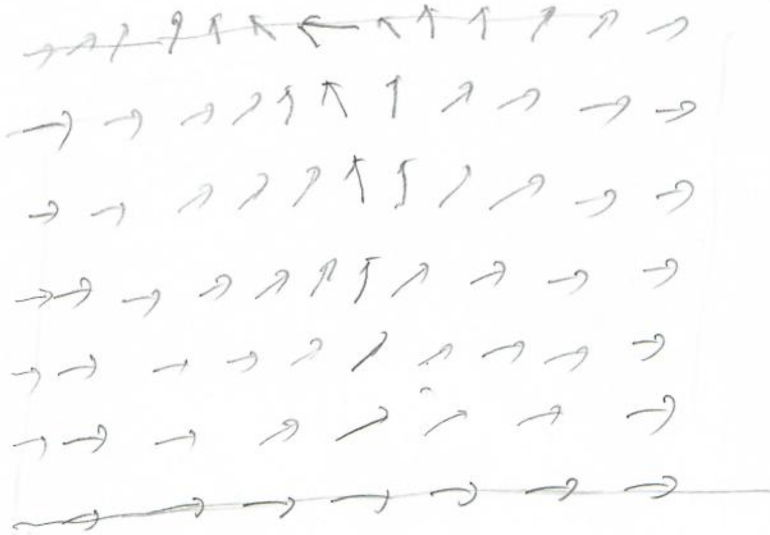
$\therefore$ For $d > 1$, $\langle e^{i\theta_0} \rangle \sim e^{i\theta - \frac{g^2}{8} \cdot \frac{S_{d-1}}{d-1} \frac{\pi^{d-1}}{(2\pi)^d}}$

$|\langle e^{i\theta_0} \rangle| < 1$, however, the expectation value of $e^{i\theta_0}$ depends on the seed state. and

There are infinitely many ground states labelled by angle θ_0.

$\Rightarrow$ $\boxed{\text{The symmetry of the Hamiltonian is spontaneously broken by the ground states.}}$

- In $d \leq 1$, the quantum fluctuations of θ are strong enough to erase all memory on the direction of the rotor (φ) in the seed state.
- The largest contributions that destroy the order come from the modes with small momenta



Infrared divergence

- In $d=0$, the ground state is unique and does not break the symmetry.

Ex)

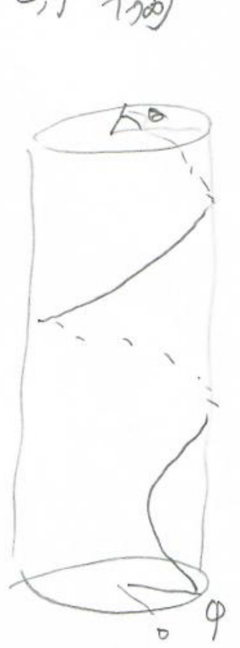
$$\Psi_{\theta,\varphi}(\theta) = \lim_{T\to\infty}\int D\theta' \, e^{-\frac{1}{g^2}\int_0^T dz\left(\frac{\partial\theta}{\partial z}\right)^2}\Bigg|_{\substack{\theta(0)=\varphi \\ \theta(T)=\theta}}$$

saddle pt: $\delta S = 0$

$$\delta S = \frac{1}{g^2}\int dz \, 2\frac{\partial\theta}{\partial z}\frac{\partial\delta\theta}{\partial z} = -\frac{2}{g^2}\int dz \, 2\left(\frac{\partial^2\theta}{\partial z^2}\right)\delta\theta = 0$$

$$\frac{\partial^2\bar{\theta}}{\partial z^2} = 0.$$

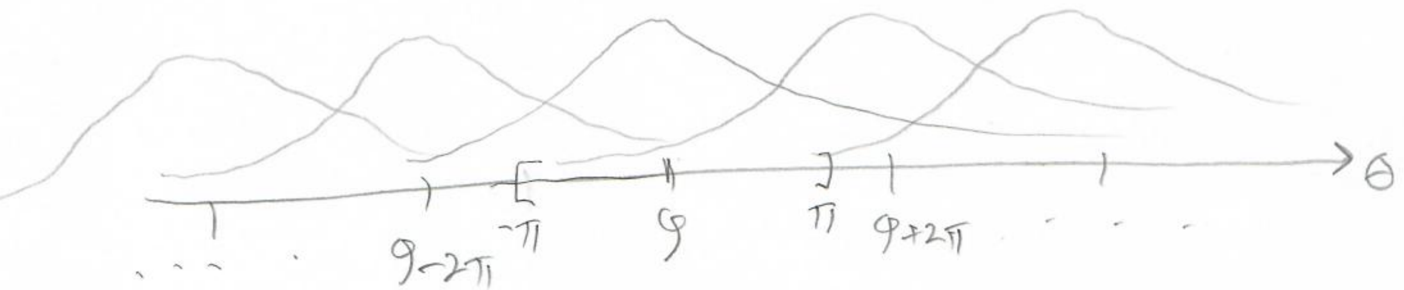
$$\bar{\theta} = az + b.$$



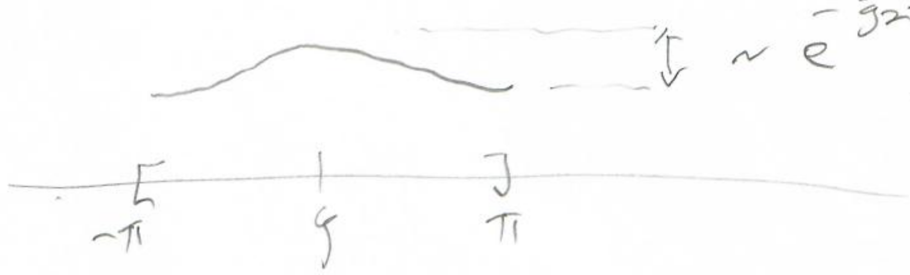
$$\theta(z) = \frac{\theta - \varphi + 2\pi m}{T} z + \theta'$$

$$S = \frac{1}{g^2}\int_0^T dz \left\{ \left(\frac{\theta - \varphi + 2\pi m}{T}\right)^2 + \left(\frac{\partial\theta'}{\partial z}\right)^2 \right\}$$

$$\Psi_{G,\varphi}(\theta) = \lim_{T\to\infty} \sum_m e^{-\frac{1}{g^2}\frac{(\theta-\varphi+2\pi m)^2}{T}} \cdot \underbrace{\left[\int D\theta' \, e^{-\frac{1}{g^2}\int_0^T \left(\frac{\partial\theta'}{\partial\tau}\right)^2}\right]_{\theta'(0)=\theta'(T)=0}}_{\text{const}}$$



$$\Downarrow$$



$$\sim e^{-\frac{1}{g^2}\frac{1}{T}} \xrightarrow[T\to\infty]{} 1$$

5. Topological order

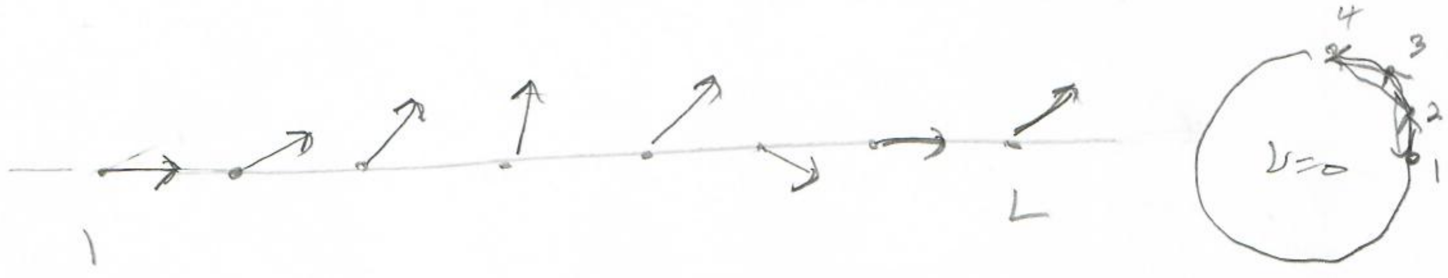
In $d=1$, the order is only marginally destroyed by the logarithmic divergence.

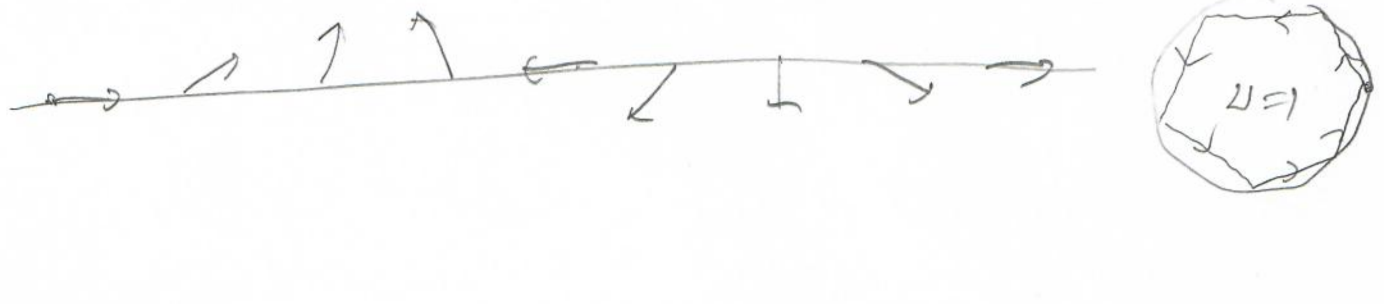
Q. does the ground state forget all memories of the seed state ?

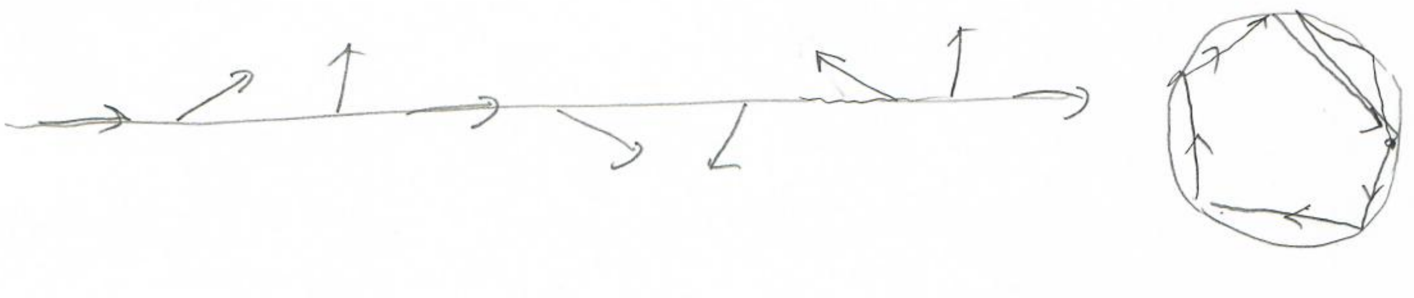
In $d=1$, there exists a global character that one can associate with each state.

winding number : $\nu = \frac{1}{2\pi} \sum_i \underline{\left[\theta_{i+1} - \theta_i\right]}$

$$-\pi \leq \left[\theta_{i+1} - \theta_i\right] < \pi$$



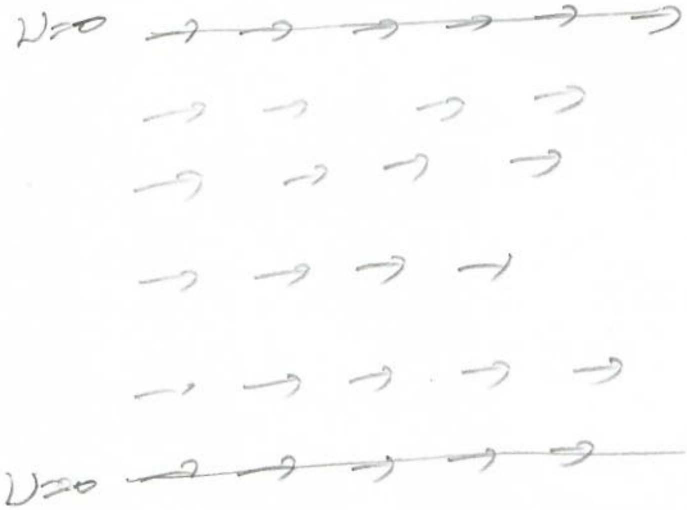




- the winding number is topological in that it is robust under a smooth deformation of the state (with smooth spin texture).
- Let's examine whether the ground state remembers the winding number of the seed state in $d=1$

$$\Psi^S_{G,0}(\theta) = \int \mathcal{D}\theta \, e^{-S} \Big|_{\substack{\theta(0)=0 \\ \theta(T)=\theta}} , \quad S = \frac{1}{g^2}\int dx\,dy\, |\nabla\theta|^2$$

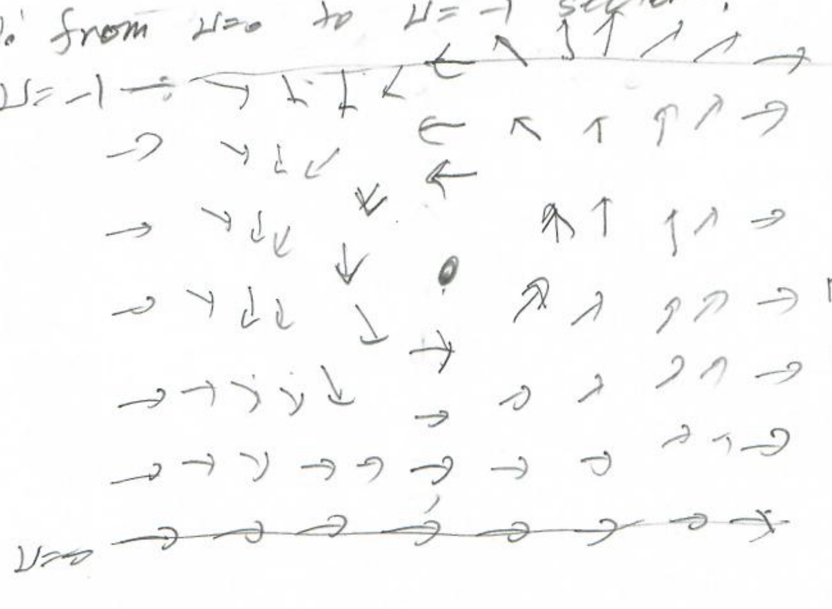
- from $\nu=0$ to $\nu=0$ sector:



$$\bar{\theta} = 0$$

$$\Psi^S_{G,0}(0) \sim 1$$

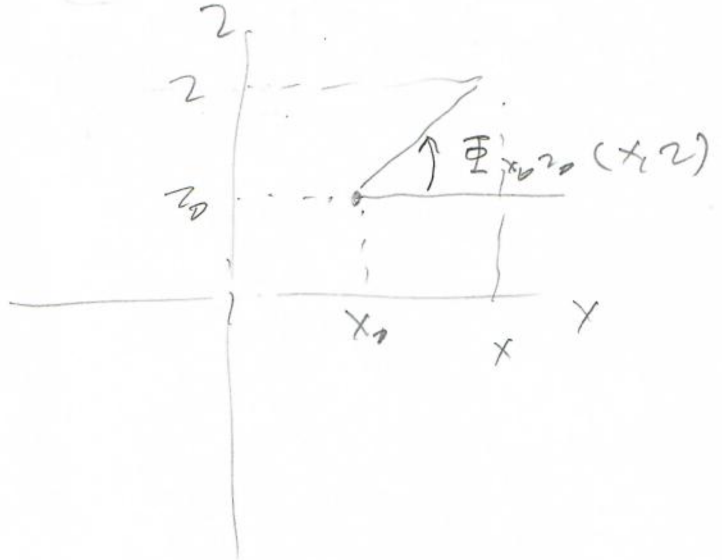
- from $\nu=0$ to $\nu=-1$ sector:



instanton describes a quantum tunneling from one winding sector to another winding sector.

An instanton action.

$$\Theta_{x_0 z_0}(x, z) = \tan^{-1} \frac{z - z_0}{x - x_0} = \Phi_{x_0 z_0}(x, z)$$



$$\nabla = \hat{r}\partial_r + \frac{\hat{\varphi}}{r}\partial_\varphi$$

$$S = \frac{1}{g^2}\int d^2\vec{r}\, |\nabla \Phi|^2 = \frac{1}{g^2}\int dr\, r\, 2\pi \left(\frac{1}{r}\right)^2$$

$$= \frac{2\pi}{g^2}\int dr \frac{1}{r} = \frac{2\pi}{g^2} \ln \frac{L}{1}$$

system size

lattice spacing.

$$\Psi_{\nu=-1, \nu=0} \sim \int dx_0 dz_0\, e^{-\frac{2\pi}{g^2}\ln L}$$

(possible position of the instanton)

$$\sim \boxed{L^2}\, e^{-\frac{2\pi}{g^2}\ln L} \sim e^{\left(2 - \frac{2\pi}{g^2}\right)\ln L}.$$

- if $g^2 > \pi$ the probability amplitude to create an instanton diverges in the $L \to \infty$ limit.

∴ G.S. won't remember the initial winding #..

- the ground state will be given by a linear superposition of states with all different winding numbers.

- if $g^2 < \pi$, the probability for quantum tunneling is suppressed in the $L \to \infty$ limit.

  $\therefore$ the GS remember the initial winding #.

$\Rightarrow$ when tunnelings from one topological sector to another topological sectors are suppressed, we say there is a "topological order"

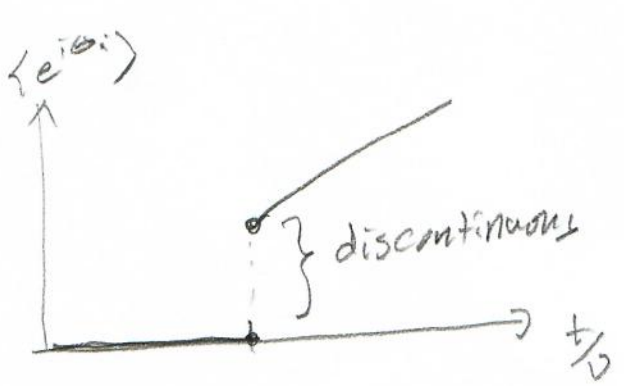
- ground state degeneracy (gapless).

6. Quantum Criticality $(d > 1)$

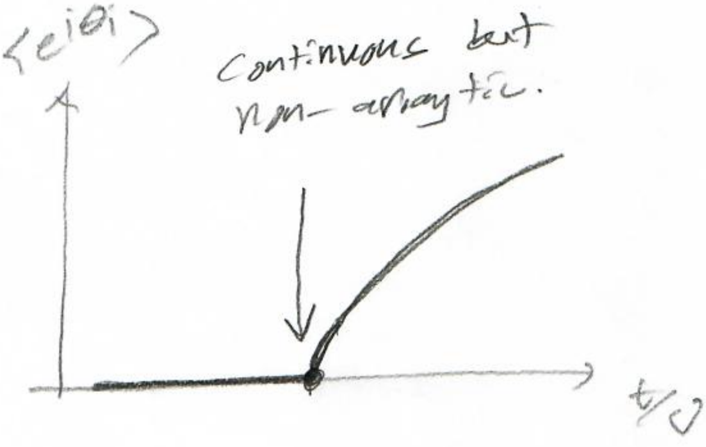
$\langle e^{i\theta_i} \rangle = 0$ unique GS.

$\langle e^{i\theta_i} \rangle \neq 0$ degenerate GS

The transition from the insulating state to the SSB state can not happen smoothly. For example, $\langle e^{i\theta_i} \rangle$ have to change non-analytically



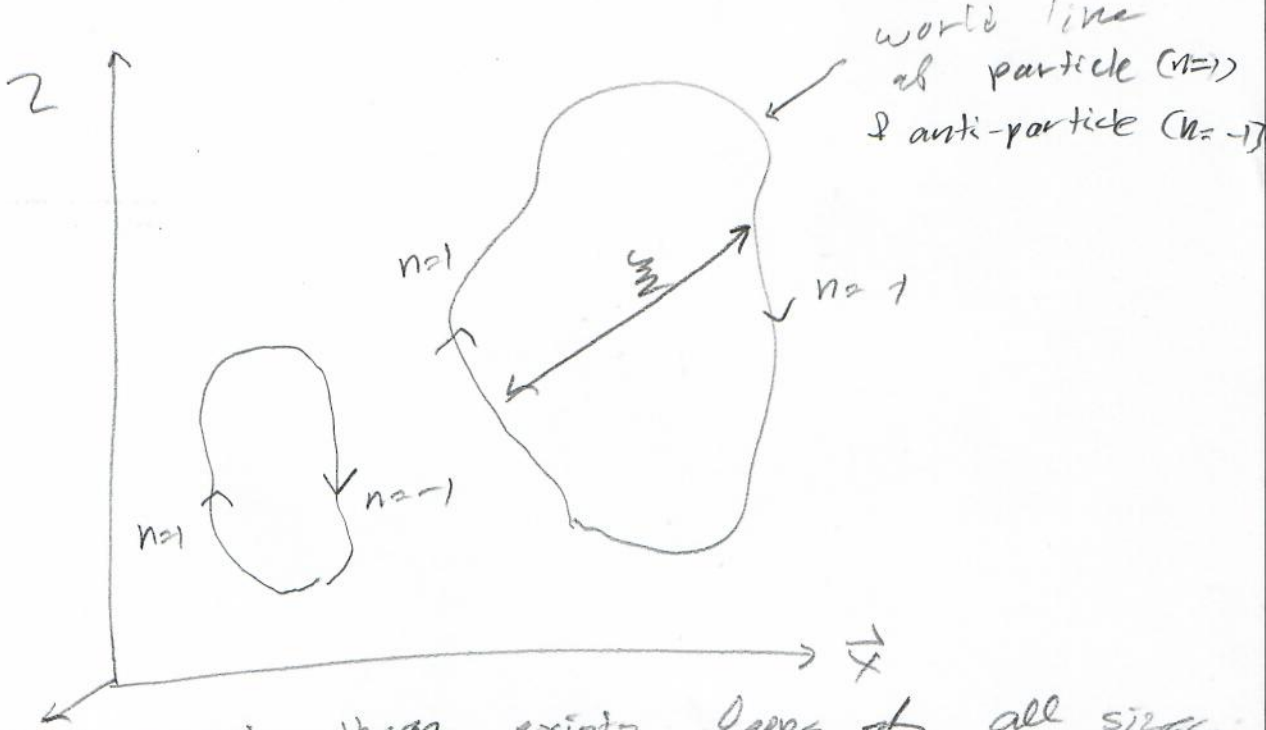
possibility 1



possibility 2

For the continuous phase transition, a new type of quantum state is realized at the critical point

As one approaches the critical point from the insulating side, the size of the particle ($n=1$) & anti-particle ($n=-1$) pairs diverges.



At the critical point, there exists loops of all sizes. And the pattern of quantum fluctuations becomes invariant under rescaling $x' = \frac{x}{b}$, $z' = \frac{z}{b}$ $(b>1)$.

∴ the critical point is described by a <u>scale invariant theory</u> that describes a strongly fluctuating "soup" of particle, antiparticle whose world lines come in all sizes

The low energy effective theory is

$$H = \int d^d\vec{x} \left[\sum_{a=1}^{2} (\hat{\pi}^a_{(x)})^2 + \sum_{a=1}^{2}\sum_{\mu=1}^{d} \partial_\mu \hat{\phi}^a_{(x)} \partial_\mu \hat{\phi}^a_{(x)} + m^2 (\hat{\phi}^2_{(x)}) + \lambda (\hat{\phi}^2_{(x)})^2 \right]$$

$$\hat{\phi}^2_{(x)} = \sum_{a=1}^{2} (\hat{\phi}^a_{(x)} \hat{\phi}^a_{(x)})$$