Gravitational Wave Astronomy exercises

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1. Derive the parsec (pc) as the distance whose annual parallax is 1".

2. Derive that 3 light years $\sim$ 1 pc.

3. Derive the "Age of the Universe" from the Hubble constant H_0 whose approximate value is $70km/sec/Mpc$, meaning that the average galaxy recession velocity is 70km/sec per each Mpc of distance.

4. Using $G_N = \frac{1\ km}{1 M_\odot}$ convert H_0^{-2} into an estimated cosmological density.

5. Compare the above result with the average density of the Universe considering that it hosts a $10^{12} M_\odot$ galaxy every Mpc^3.

6. Under the blunt approximation of constant matter density inside a galaxy (!), estimate its typical rotational velocity ω at a distance of $\sim$ 10kpc from its center. How does it change with the distance from the galactic center?

7. Use the Planck constant value $\hbar \simeq 200$ MeV$\times$ fm (and $1eV \simeq 1.8 \times 10^{-33} gr$) to derive

$$M_{Pl} \equiv \left(\frac{\hbar}{G_N}\right)^{1/2} \simeq 10^{-5} gr \simeq 10^{19} GeV\,,$$

   and that $M_\odot \simeq \left(\frac{M_{Pl}}{m_p}\right)^2 M_{Pl} \simeq 10^{33}$gr. The reason of this last equality (two largely different microscopic quantities conjuring to a macroscopic one) reside on the equilibrium configuration of a star sustained by zero-degeneracy pressure, a subject adressed by Tolman-Oppenheimer-Volkoff equation in General Relativity.

8. Verify that the homogeneous Maxwell's equations

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0\,, \\ \vec{\nabla} \times \vec{E} + \tfrac{\partial \vec{B}}{\partial t} &= 0\,, \end{aligned}$$

are identically satisfied if one assumes $E^i = F^{0i}$, $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$, with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\mu A_\nu$ (metric convention -+++).

9. Verify that the in-homogeneous Maxwell's equations

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 4\pi\rho\,, \\ \vec{\nabla} \times \vec{B} &= \tfrac{\partial \vec{E}}{\partial t} + 4\pi\vec{J}\,, \end{aligned}$$

are equivalent to

$$\partial_\nu F^{\nu\mu} = 4\pi J^\mu\,.$$

10. Show that the decomposition $A_i = \bar{A}_i + \partial_i a$ admits a unique solution for a and $\bar{A}_i$ in terms of the original A_i (assuming all go to 0 at infinity) under the condition $\partial_i \bar{A}^i = 0$. How many degrees of freedom are encoded in each of $A_i, \bar{A}_i$, and a? What are their dimensions?

11. Derive the e.o.m. for $\bar{A}^i$: $\Box \bar{A}^i = 4\pi \bar{J}^i$, where $J^i = \bar{J}^i + \partial^i j$. Interpret this equation as a wave equation.

12. Show that the remaining two non-homogeneous Maxwell's equations are equivlent iff $\dot{\rho} + \nabla^2 j = 0$, and that they both lead to

$$\nabla^2 \left(A^0 + \dot{a}\right) = 4\pi\rho\,.$$

Show that this implies that the field $A^0 + \dot{a}$ does not propagate, but it is a d.o.f. dependent on the source ρ.

13. Verify that $P_{ij}(\hat{n}) \equiv \delta_{ij} - \hat{n}_i\hat{n}_j$ is a projector, i.e. $P^2 = P$. Veirfy that it projects onto the space orthogonal to $\hat{n}$.

14. Verify that the retarded and advanced Green's functions can be written as

$$G_R(t-t', \mathbf{x}-\mathbf{x}') = -\frac{\delta(t - |\mathbf{x}-\mathbf{x}'|)}{4\pi|\mathbf{x}-\mathbf{x}'|} = \int \frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3}\frac{e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}}{(\omega+i\epsilon)^2 - k^2}\,.$$

When integrating $\int_0^\infty e^{i\omega r}$ it is crucial to replace $\omega \to \omega + i\epsilon$ to make the integral insensitive to $r \to \infty$ features of the source. Analogously for advanced boundary condition one has to substitute $\omega \to \omega - i\epsilon$.

15. Locate the position of the poles of the retarded and advanced Green's function in the complex ω plane.

16. In particle physics one adopts a different boundary coundition, giving rise to the Feynman Green's function

$$\tilde{G}_F = \frac{1}{\omega^2 - \mathbf{k}^2 + i\epsilon}\,,$$

whose poles in the complex k plane are for $k^2 = \omega^2 + i\epsilon$, i.e. $k = \omega + i\epsilon$ and $k = -\omega - i\epsilon$ for $\omega > 0$ and $k = -\omega + i\epsilon$ and $k = \omega - i\epsilon$ for $\omega < 0$, i.e. in a unique form: $k = \pm\left(|\omega| + i\epsilon\right)$.
Perform the space k integral first in

$$G_F(t,x) = \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{e^{-i\omega t + ikx}}{\omega^2 - k^2 + i\epsilon}\,,$$

to obtain

$$G_F \sim \frac{1}{r} \int d\omega e^{-i\omega t + i|\omega| r}\,,$$

which shows that for $\omega > 0$ (< 0) one has out-(in-)going waves.

17. Verify that

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}(\hat{n})P_{jl}(\hat{n}) - \frac{1}{2}P_{ij}(\hat{n})P_{kl}(\hat{n})\,,$$

is a projector acting a symmetric rank 2 tensor h_{ij}, given that

$$P_{ij} \equiv \delta_{ij} - \hat{n}_i\hat{n}_j$$

is the transverse projector acting on vectors. Show that the outcome of the projection

$$h_{ij}^{TT} = \Lambda(\hat{n})_{ij,kl} h_{kl}\,,$$

is a trace-less tensor with no components along $\hat{n}$.

18. Show that for a wave propagating along $\hat{z}$ the radiative metric perturbation can be written as

$$h_{ij}^{TT} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}\,,$$

hence it indeed has 2 radiative degrees of freedom.

19. Given the GW equation

$$\Box h_{ij}^{TT} = -16\pi G \Lambda_{ij,kl} T_{kl}$$

derive the solution

$$h_{ij}^{TT}(t, r\hat{n}) = 4G\Lambda_{ij,kl}(\hat{n}) \int \frac{T(t-|x-x'|, x')}{|x-x'|} d^3x' \,.$$

20. Show that for small sources with small internal velocities one can approximate the previous solution as

$$h_{ij}^{TT}(t, r\hat{n}) = \Lambda_{ij,kl}(\hat{n}) \frac{4G}{r} \int d^3x' \left[T_{kl}(t-r, x') - \dot{T}_{kl}(t-r, x') \vec{x}' \cdot \hat{n} + \ldots \right] .$$

What is the expansion parameter?

21. Given the energy-momentum tensor of GWs

$$T_{\mu\nu}^{(GW)} = \frac{1}{32\pi G_N} \langle \partial_\mu h_{ij}^{TT} \partial_\nu h_{ij}^{TT} \rangle$$

derive the energy density in GWs

$$\rho_{GW} = \frac{1}{16\pi G_N} \left(\dot{h}_+^2 + \dot{h}_\times^2 \right) , \tag{1}$$

and the radial outgoing flux $T^{0r} = \rho$.

22. Using the GW e.o.m. $\Box h_{ij}^{TT} = -16\pi G_N \Lambda_{ij,kl} T_{ij}$, derive the luminosity formula

$$\frac{dE}{dt} = \frac{G}{8\pi} \dddot{Q}_{ij} \dddot{Q}_{kl} \int \Lambda_{ij,kl}(\hat{n}) d\Omega \,.$$

Use

$$\begin{aligned} \frac{1}{4\pi} \int \hat{n}^i \hat{n}^j d\Omega &= \frac{\delta^{ij}}{3} \,, \\ \frac{1}{4\pi} \int \hat{n}^i \hat{n}^j \hat{n}^k \hat{n}^l d\Omega &= \frac{1}{15} \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right) \,, \end{aligned}$$

to derive the quadrupole formula

$$\frac{dE}{dt} = \frac{G}{5} \dddot{Q}_{ij} \dddot{Q}_{kl} \left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl} \right) .$$

23. Specialize the quadrupole to a binary system in circular orbit

$$\begin{aligned} x &= r\cos(\omega t) \,, \\ y &= r\sin(\omega t) \,, \\ z &= = 0 \,, \end{aligned}$$

to derive

$$\begin{aligned}
\dddot{Q}_{xx} &= 4\eta M\omega^3 r^2 \sin(2\omega t)\ , \\
\dddot{Q}_{xy} &= -4\eta M\omega^3 r^2 \cos(2\omega t)\ , \\
\dddot{Q}_{yy} &= -4\eta M\omega^3 r^2 \sin(2\omega t)\ ,
\end{aligned}$$

from which

$$\dot{E} = \frac{32G}{5}\eta^2 M^2\omega^6 r^4 = \frac{32}{5G}\eta^2 v^{10}\,.$$

where in the last passage the Kepler's law has been used (hence while the quadrupole formula holds for generic system, the expression above involving v^{10} holds only for self-gravitating ones.)

24. Compute the amplitude of gravitational wave emitted by a quadrupolar, planar source for generic propagation direction via

$$h_{ij}^{TT}(\hat{n}) = \Lambda_{ij,kl}(\hat{n})\frac{2G}{r}\begin{pmatrix} \ddot{Q}_{xx} & \ddot{Q}_{xy} & 0 \\ \ddot{Q}_{xy} & -\ddot{Q}_{xx} & 0 \\ 0 & 0 & 0 \end{pmatrix}\,,$$

by using the simplified setup of $\hat{n} = \hat{z} = (0,0,1)$ and tilting the source by $-\iota$:

$$\begin{aligned}
h_{ij}^{TT}(\hat{z},\iota) = \Lambda_{ij,kl}(\hat{z})\frac{2G}{r}&\begin{pmatrix} \cos\iota & 0 & -\sin\iota \\ 0 & 1 & 0 \\ \sin\iota & 0 & \cos\iota \end{pmatrix}\begin{pmatrix} \ddot{Q}_{xx} & \ddot{Q}_{xy} & 0 \\ \ddot{Q}_{xy} & -\ddot{Q}_{xx} & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\iota & 0 & \sin\iota \\ 0 & 1 & 0 \\ -\sin\iota & 0 & \cos\iota \end{pmatrix} \\
&= \frac{2G}{r}\begin{pmatrix} \frac{Q_{xx}}{2}(1+\cos^2\iota) & Q_{xy}\cos\iota & 0 \\ Q_{xy}\cos\iota & -\frac{Q_{xx}}{2}(1+\cos^2\iota) & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

25. Derive the "Newtonian" evolution of the binary circular velocity:

$$\frac{dv}{v^9} = \frac{32\eta}{5G_N M}dt \Longrightarrow \frac{\Delta t}{GM} = \frac{5}{256\eta}\left(\frac{1}{v_i^8} - \frac{1}{v_f^8}\right)\,.$$

26. Derive the evolution of the GW frequency using that $v = (\pi G M f_{GW})^{1/3}$:

$$\begin{aligned}
\dot{f}_{GW} &= \frac{96}{5}\pi^{8/3}\eta\,(GM)^{5/3} f_{GW}^{11/3} \\
&= \frac{96}{5}\pi^{8/3}(GM_c)^{5/3} f_{GW}^{11/3}\,.
\end{aligned}$$

27. It is convenient to express the Fourier Transform of the waveform via the analytic formula ($\iota = 0$)

$$\tilde{h}(f) = \eta \frac{GM}{r} \int dt v^2(t) e^{2i\pi f t} \left(e^{i\phi(t)} + e^{-i\phi(t)} \right) ,$$

using the *Stationary Phase Approximation* leading to

$$\begin{aligned} \tilde{h}(f) &\simeq \eta \frac{GM}{r} v^2(t_0(f)) e^{2\pi i f t_*(f)} \int e^{-i\phi_* - i\ddot{\phi}(t-t_*)^2} dt \\ &\simeq \frac{\pi^{2/3} (GM_c)^{5/3}}{r} f^{2/3} \left(\frac{2\pi}{\dot{f}} \right)^{1/2} e^{i(2\pi f t_0 - \phi_* - \pi/4)} \\ &\simeq \pi^{-2/3} \left(\frac{5}{24} \right)^{1/2} \frac{(GM_c)^{5/6} f^{-7/6}}{r} e^{i(2\pi f t_* - \phi(t_*) - \pi/4)} . \end{aligned}$$

28. Derive the numerical relation

$$\Delta t \simeq \frac{5 G_N M}{256 \eta} \simeq 1.4 \times 10^4 sec \ (\eta)^{-1} \left(\frac{M}{M_\odot} \right) \left(\frac{f_{iGW}}{10Hz} \right)^{-8/3} ,$$

using $v = (\pi G M f_{GW})^{1/3}$

29. Given the detector output $n(t)$ (assuming it is all noise), consider its discrete Fourier Transform

$$\tilde{n}(f_k) = \Delta f \sum_j e^{2\pi i f_k t_j} n(t_j) ,$$

where $f_k = k\Delta f$, $\Delta f = 1/T$, being $T = N\Delta t$ the total acquisition time, N the total number of points sampled and Δt the sampling time. Show that while the amplitude of $\tilde{n}(f)$ depends on N, the spectral noise density S_n defined by

$$\langle \tilde{n}(f) \tilde{n}(f') \rangle = S_n \delta(f + f') ,$$

or its discrete analog

$$\langle \tilde{n}(f_j) \tilde{n}^*(f_k) \rangle = S_n \delta_{jk} \frac{1}{\Delta f} ,$$

is independent on N, hence it represents a good figure of merit of the detector's noise. What is S_n dimension?

30. To assess the strength of a signal, it is necessary to compare it to the detector's noise. Let us then consider *filtering* the signal by correlating to a filter $K(t)$ and comparing this to the noise:

$$\frac{S}{\sqrt{N^2}} = \frac{\int dt s(t)K(t)}{\left[\int dtdt' K(t)K(t')\langle n(t)n(t')\rangle\right]^{1/2}}$$
$$= \frac{\int df \tilde{s}(f)\tilde{K}^*(f)}{\sqrt{\int df S_n |K(f)|^2}^{1/2}} .$$

To find the best possible filter K define the scalar product

$$\langle A|B\rangle \equiv \int df \frac{(AB^* + A^*B)}{S_n} ,$$

hence

$$\frac{S}{\sqrt{N^2}} = \frac{\langle \tilde{s}|\tilde{u}\rangle}{\langle \tilde{u}|\tilde{u}\rangle^{1/2}} ,$$

with $\tilde{u} \equiv S_n\tilde{K}$. The solution can only be $\tilde{K} = \tilde{s}/S_n$ (why?) hence giving

$$SNR = \left[2\int df \frac{|\tilde{h}(f)|^2}{S_n}\right]^{1/2} .$$