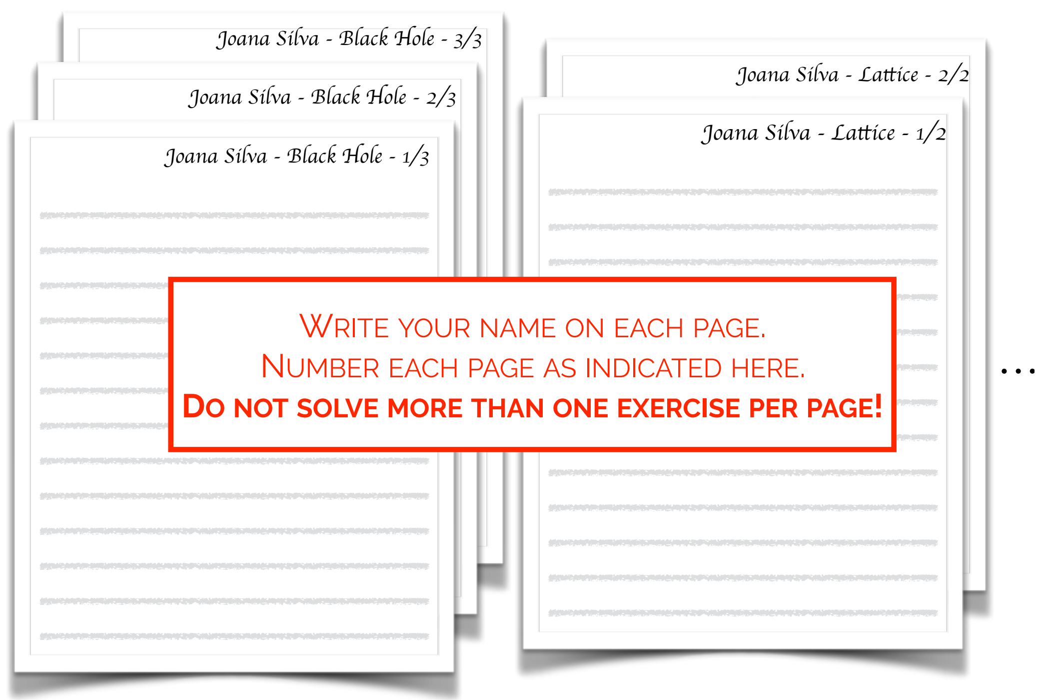
IFT-Perimeter-SAIFR

Journeys into Theoretical Physics

2017

Morning Exam

Important:



Scores:

- Problem 1 (The Big Rip): 25%
- Problem 2 (Linear Chains of Atoms): 25%
- Problem 3 (Falling into a Black Hole): 25%
- Problem 4 (Axions and the ADMX Experiment): 25%

1 The Big Rip

Consider a Universe today that is spatially flat (implying that its density is the critical density) composed of 2 fluids: non-relativistic matter and a dark energy fluid with equation of state $P_{DE} = \omega\rho_{DE}$ with constant ω.

1. **[3 pts]** Show that its energy density is proportional to

$$\rho_{DE} \propto a^{-3(1+\omega)} \tag{1}$$

[You can use, eg, conservation of energy: $dU = -PdV$, with $U = \rho V$ and a simple scaling relation between the volume and the scale factor.]

2. **[3 pts]** Show that Friedmann's 1st equation can be written as:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_m a^{-3} + (1-\Omega_m)a^{-3(1+\omega)}\right] \tag{2}$$

where

$$H_0^2 = \frac{8\pi G}{3}\rho_{cr}^{(0)} \tag{3}$$

and

$$\Omega_i = \frac{\rho_i^{(0)}}{\rho_{cr}^{(0)}} \tag{4}$$

3. **[8 pts]** Show that for this Universe with $\omega = -1-\delta$ with $\delta > 0$ (sometimes called phantom dark energy) one encounters a singularity in the future where $a \to \infty$ (this is the so-called big rip singularity) at a finite time (t_{rip}). In the case where the Universe is dominated by this fluid find an expression for ($t_{rip} - t_0$), with t_0 being today.

4. **[3 pts]** The contribution to the acceleration of a test particle due to a dark energy fluid can be obtained by noticing that a physical distance x can be written in terms of a comoving (constant) distance r using the scale factor: x = a r. Show that

$$\ddot{x} = \frac{\ddot{a}}{a}x = -\frac{4\pi G}{3}(\rho_{DE} + 3P_{DE})x \tag{5}$$

5. **[8 pts]** In these phantom models the contribution of dark energy grows with time, leading to the disruption of bound systems before the big rip. Show that for a planet orbiting a star of mass M at a radius R this disruption will happen when

$$a = \left[\frac{3M}{4\pi R^3}\frac{1}{(2+3\delta)\rho_{DE}^{(0)}}\right]^{\frac{1}{3\delta}} \tag{6}$$

2 Linear Chains of Atoms

Consider a one-dimensional periodic chain of N atoms as shown in figure 1 – the last atom in the chain connects back to the first one. We use a for the separation between neighbooring sites. As a first approximation, each isolated atom i is described by a single energy level ϵ_i and its corresponding eigenstate $|i\rangle$. They interact via a general coupling constant γ_{ij} describing the interaction between sites i and j.

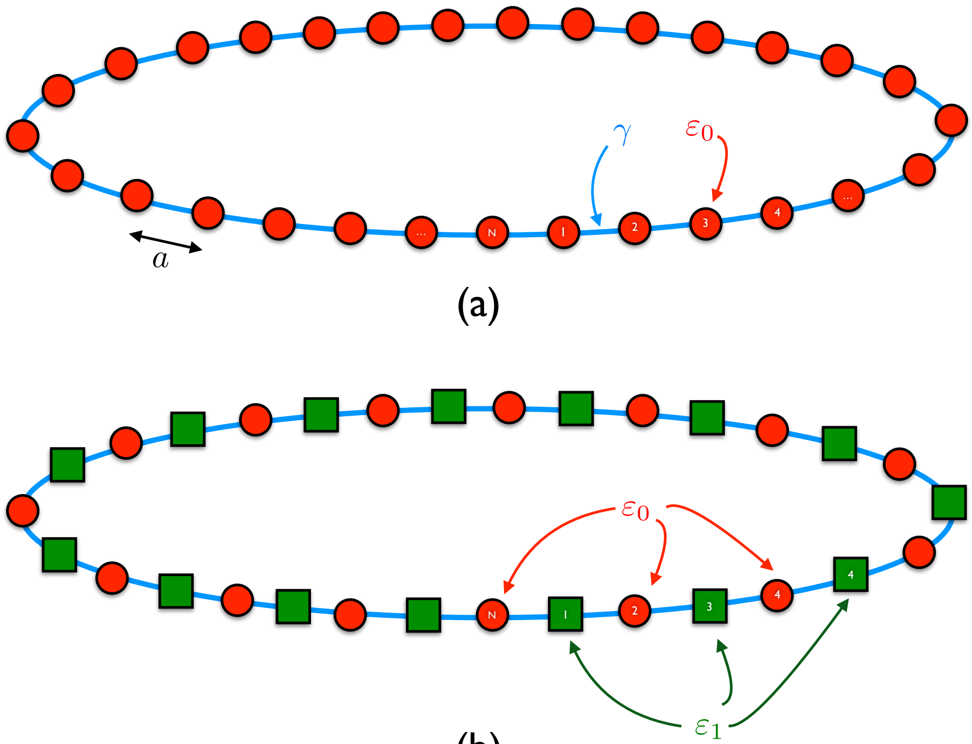


Figure 1: Example of two periodic chains with N atoms. **Top:** Homogenous chain with onsite energy ε_0 and nearest neighbour interaction γ. **Bottom:** Alternating chain with alternating onsite energy $\varepsilon_0/\varepsilon_1$ and nearest neighbour interaction γ.

Homogeneous Chain

Assuming only coupling between nearest neighbors,

$$\gamma_{ij} = \begin{cases} \gamma & \text{for } j = i \pm 1 \\ 0 & \text{otherwise} \end{cases} , \tag{7}$$

and $\langle i\,|j\rangle = \delta_{ij}$ (states between adjacent sites are orthogonal), the Hamiltonian for the this problem can be written as

$$\mathcal{H} = \sum_i \epsilon_i \,|i\rangle \langle i| + \sum_i \gamma \,|i\rangle \langle i \pm 1| \tag{8}$$

1. **[3 pts]** If $\epsilon_i = \epsilon_0 \; \forall i$ (all atoms are identical), show that

$$|\Psi_k\rangle = \frac{1}{\sqrt{N}} \sum_j e^{ijka} |j\rangle \ , \tag{9}$$

   with $i = \sqrt{-1}$, are solutions to the problem $\mathcal{H}|\Psi\rangle = E|\Psi\rangle$.

2. **[3 pts]** What are the possible values of k, and what are the corresponding energy eigenvalues?

Alternating Chain

3. **[5 pts]** What would happen to the distribution of eigenvalues if one considered alternating onsite energies, *i.e.*

$$\epsilon_i = \begin{cases} \epsilon_0 & \text{for } i \text{ even} \\ \epsilon_1 & \text{for } i \text{ odd} \end{cases} \ , \tag{10}$$

   as indicated in figure 1b. Is there a gap in the distribution, and if so, what would be its value?

Chain with connected atom

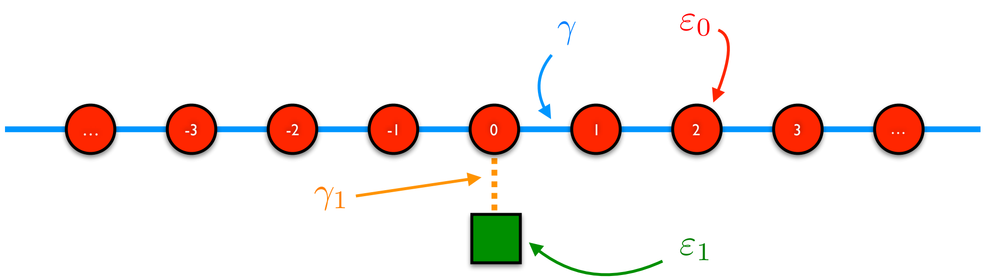


Figure 2: A single atom attaches to an homogenous chain.

Consider now the situation depicted in figure 2 where a single atom with onsite energy ϵ_1 and coupling constant γ_1 is connected to an infinite homogenous chain.

4. **[4 pts]** Write down the Hamiltonian for this problem.

5. **[5 pts]** Write down a proposed single particle eigenstate for this Hamiltonian.

6. **[5 pts]** Solve for this proposed eigenstate, and calculate the Büttiker-Landauer conductance as a function of energy.

3 Falling Into a Black Hole

1. [**4 pts**] The following warm-up integrals might be useful below. Show that

$$\int_0^1 \frac{dx}{x\sqrt{\frac{1}{x}-1}} = \pi\,, \qquad \int_0^1 \frac{dx}{\sqrt{\frac{1}{x}-1}} = \frac{\pi}{2}\,. \tag{11}$$

   Hint: These are rather easy to compute using simple complex analysis but **if you don't see it right away, just take them as given and move on to the more important physics part of the question**. You can come back to computing these integrals if you have time at the end of the examination period.

Recall the Schwarzschild metric in geometrized units:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,.$$

An observer (Alice) falls from the horizon at $r = 2M$ to the singularity at $r = 0$ along a radial trajectory (all other coordinates are fixed).

2. [**7 pts**] How much time passes on her clock along this trajectory?

3. [**6 pts**] Show that this is the maximum time that any observer will have between crossing the horizon and meeting the singularity. That is, an observer may have a powerful rocket engine but no matter what trajectory she tries to follow, the time will always be shorter than (or equal to) the result in the first part of the question.

4. [**8 pts**] As Alice starts falling, she emits a photon in the ϕ direction. As she falls towards the singularity, how many times will she pass by that emitted photon? In other words, which of of the dashed curves in figure 3 best describes the photon trajectory in its infall towards the singularity?

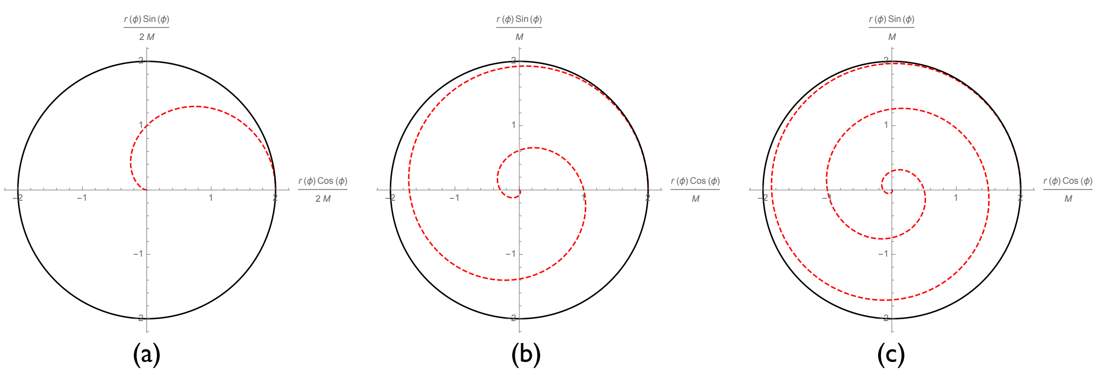


Figure 3: Hypothetical photon trajectories (red dashed curves). Which one is correct?

4 Axions and the ADMX Experiment

An axion $\alpha(\vec{x},t)$ can couple to electromagnetism through

$$\kappa\ \alpha(\vec{x},t)\vec{E}\cdot\vec{B}$$

In the presence of such a coupling, Maxwell's equations are modified as follows:

$$\nabla\cdot\vec{E} = \rho - \kappa\ \nabla\alpha\cdot\vec{B} \tag{12}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{13}$$

$$\nabla\cdot\vec{B} = 0 \tag{14}$$

$$\nabla\times\vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{j} + \kappa\ (\dot{\alpha}\vec{B} + \nabla\alpha\times\vec{E}) \tag{15}$$

This coupling mixes the axion and the photon in such a way that a background magnetic or electric field in the presence of an axion will generate a small magnetic and or electric field that can later be detected with a sensitive detector as illustrated in this exercise. This is the idea behind the ADMX experiment.

Note: In the questions below you will often find justifiable to drop spacial gradients or time derivatives, depending on the situations. Also, although exact solutions to the equations are welcome, solutions that are approximate and correct at the order of magnitude level are acceptable.

1. **[5 pts]** Take a constant electric field E_0 (and zero initial magnetic field) applied in a region of space with size $R \gg m_\alpha^{-1}$, where m_α is the axion mass. This field is generated in a region where there is an axion field configuration $\alpha(x,t) = \alpha_0 e^{-im_\alpha t}$. Does the presence of the axion induce any (time dependent) electric or magnetic field in the presence of E_0? If yes, compute those induced fields.

2. **[5 pts]** Consider now a constant magnetic field B_0 (and zero initial electric field) applied again in a region of space with size $R \gg m_\alpha^{-1}$ with an axion field configuration $\alpha(x,t) = \alpha_0 e^{-im_\alpha t}$. Does the presence of the axion induce any (time dependent) electric or magnetic field in the presence of B_0? If yes, compute the induced fields.

3. **[5 pts]** If the axion is the Dark Matter, it can be thought of as a monochromatic wave with a momentum $m_\alpha v_{vir}$with respect to the earth, where $v_{vir} \sim 10^{-3}$ is the virial velocity. How does the answer in the previous two points change? (Hint: Since the virial velocity is small, only one of the points above is significantly affected.) Given your findings, what gives the biggest signal for an axion Dark Matter search: turning on an electric or a magnetic field?

4. **[10 pts]** Consider now the case where the region where the magnetic or electric field is applied is much smaller than the axion Compton wavelength. What is the biggest magnetic or electric field induced for axion Dark Matter searches in the optimal of the two scenarios?