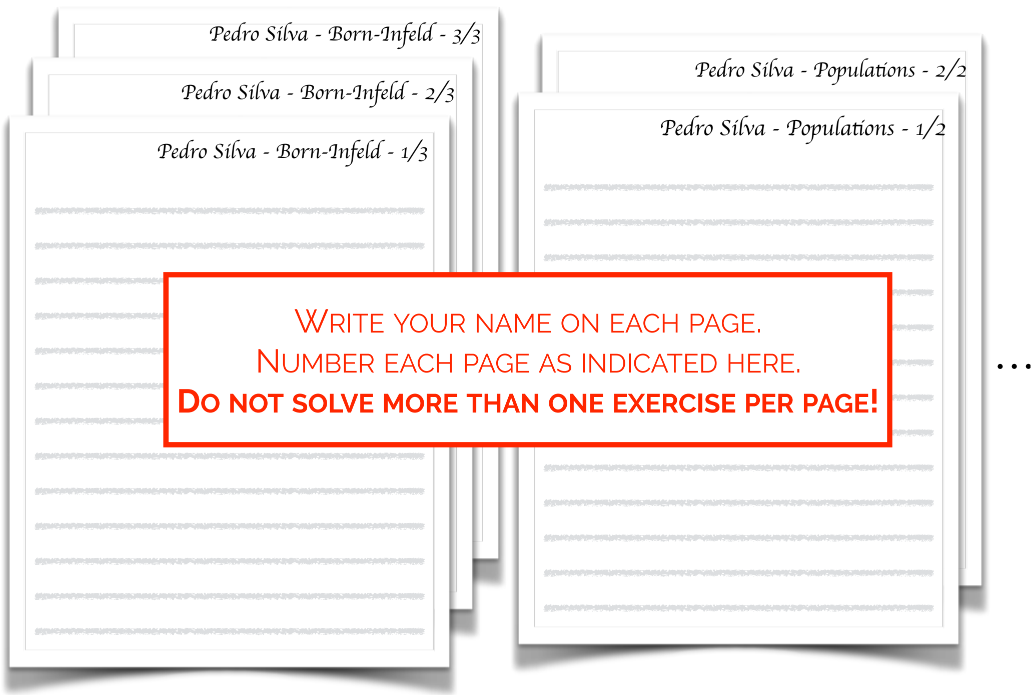
IFT-Perimeter-SAIFR

Journeys into Theoretical Physics

2017

Afternoon Exam

Important:



Scores:

- Problem 1 (Born-Infeld Electromagnetism): 25%
- Problem 2 (Restricted Three-Body Problem): 25%
- Problem 3 (Biological Population Dynamics): 25%
- Problem 4 (An Inverse Problem in Quantum Mechanics): 25%

Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 Born-Infeld Electromagnetism

In Maxwell's theory for electromagnetism, a point charge generates an electric field which carries infinite energy. So the self-energy of a point charge in this theory is infinite. In 1934, Born and Infeld developed a modified theory of electromagnetism in which a point charge generates an electric field which carries finite energy. The first part of this question includes warm-up exercises about Maxwell theory, and the second part of the question concerns properties of the Born-Infeld theory of electromagnetism.

We will use conventions where the speed of light $c = 1$ and space-time signature corresponding to $\eta^{mn} = \text{diag}(+1, -1, -1, -1)$. Recall that

$$\vec{E} = -\vec{\nabla}\phi - \frac{d\vec{A}}{dt}\,, \quad \vec{B} = \vec{\nabla} \times \vec{A},$$

can be encoded in relativistic notation simply as $F_{mn} = \frac{d}{dx^m}A_n - \frac{d}{dx^n}A_m$ where $m = 0...3$ and $A^m = (\phi, \vec{A})$. More precisely, we have $F_{0j} = E_j$ and $F_{jk} = -\epsilon_{jkl}B^l$ where here the indices run over space values only $j = 1...3$. Two relativistic formulas which can easily be verified are

$$F_{mn}F^{mn} = -2(\vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B})\,, \quad \epsilon_{mnpq}F^{mn}F^{pq} = -8\vec{E}\cdot\vec{B}$$

Finally, when discussing matter sources we will also pack them into a four vector in the standard way as $j^m = (\rho, \vec{j})$.

Electromagnetism Warm-Up

1. **[2 pts]** Show that the relativistic equation

$$\eta^{mn}\frac{d}{dx^m}F_{np} = j_p \tag{1}$$

is implied by Maxwell's equations when expressed in terms of $\vec{E}$ and $\vec{B}$ in the presence of sources ρ and $\vec{j}$.

2. **[1 pts]** Show that Maxwell's equations imply that the electric field produced by a static point charge Q localized at the origin $x = y = z = 0$ is

$$\vec{E} = \frac{Q}{4\pi}\frac{\vec{R}}{|R|^3}$$

where $\vec{R} = (x, y, z)$ and $|R| = \sqrt{x^2 + y^2 + z^2}$.

3. **[1 pts]** Show that the total energy of this electric field, $\mathcal{E} = \frac{1}{2}\int d^3x\, \vec{E}\cdot\vec{E}$, is infinite.

4. **[2 pts]** Show that equation (1) follows from extremizing the action

$$S = \int d^4x[\frac{1}{4}F_{mn}F^{mn} + A_m j^m]$$

with respect to variation of the gauge field A_m.

Born-Infield

5. **[2 pts]** For a general action of the form

$$S = \int d^4x[f(\vec{E}, \vec{B}) + A^m j_m]$$

where f is an arbitrary function of the electric and magnetic fields, show that extremizing the action with respect to A_m implies the equations of motion

$$-\frac{d}{dt}\vec{D} + \vec{\nabla} \times \vec{H} = \vec{j}, \quad \vec{\nabla} \cdot \vec{D} = \rho$$

where $\vec{D} \equiv -\frac{\partial f}{\partial \vec{E}}$ and $\vec{H} \equiv \frac{\partial f}{\partial \vec{B}}$.

6. **[2 pts]** In the action proposed by Born and Infeld,

$$f(\vec{E}, \vec{B}) = b^2 \sqrt{1 + \frac{1}{2b^2} F_{mn}F^{mn} - (\frac{1}{8b^2}\epsilon^{mnpq}F_{mn}F_{pq})^2} - b^2$$

where b is a constant. Show that Born-Infeld theory is equivalent to Maxwell theory in the limit that $b \to \infty$.

7. **[2 pts]** Write the equations of motion in terms of $\vec{E}$ and $\vec{B}$ that come from extremizing the Born-Infeld action with respect to A_m. The formulas in the introduction may be useful.

8. **[5 pts]** Find the electric field in Born-Infeld theory produced by a static point charge Q localized at the origin. Hint: First find the expressions for $\vec{D}$ and $\vec{H}$.

9. **[2 pts]** Show that the electric field has a maximum value of $|E| = b$. Sketch the magnitude $|E|$ as a function of the distance R for $b = 1$ and compare it with the usual EM case corresponding to $b \to \infty$.

10. **[3 pts]** Show that the total energy $\mathcal{E}$ of the electric field in Born-Infeld theory is finite where the total energy of the electric field is defined by

$$\mathcal{E} = \int d^3x(\vec{D} \cdot \vec{E} + f(\vec{E}, \vec{B})).$$

11. **[3 pts]** Show that the Born-Infeld action can be concisely expressed as

$$f(\vec{E}, \vec{B}) = b^2 \sqrt{-\det(\eta_{mn} + \frac{1}{b}F_{mn})} - b^2$$

In your argument, you can use the fact that at any point, one can always choose a Lorentz frame such that the only nonzero components of F_{mn} are $F_{10} = -F_{01} = |E|$ and $F_{23} = -F_{32} = |B|$, i.e. $\vec{E}$ and $\vec{B}$ are both pointing in the x direction.

2 Restricted Three-body Problem

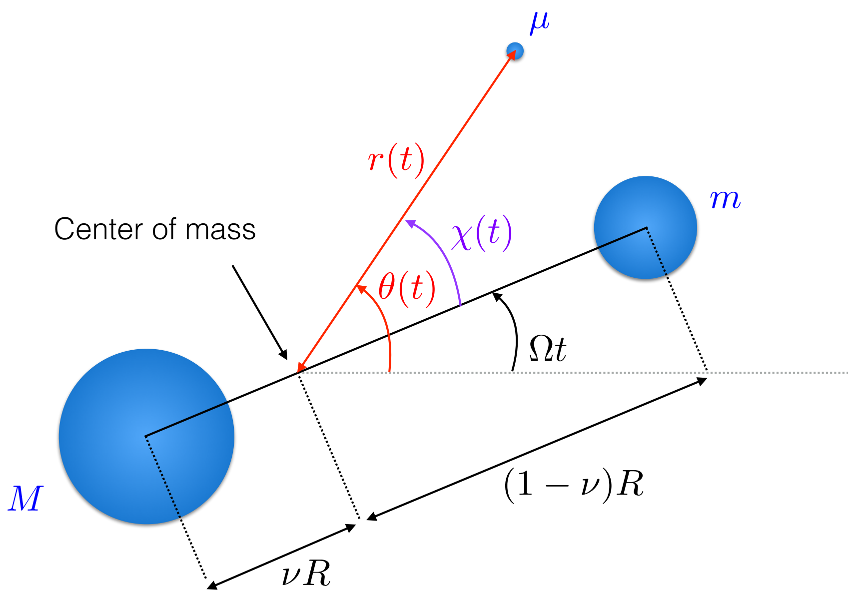


Figure 1: A probe object orbits around two much heavier objects. In this exercise we will look for equilibrium points of this probe particle which can be nice locations to put satellites, when they are stable.

Consider two heavy masses $M > m$ in circular orbit around each other with angular frequency Ω at a constant distance of R away from each other. The restricted three-body problem is to determine the motion of a light object of negligible mass μ in the gravitational field of the two heavy masses. We set $\mu = 1$ for convenience and assume $M, m \gg 1$. We also assume all motion to take place in the same plane. Prototypical examples could be the Moon moving in the gravitational field of the Sun and the Earth or a small satellite moving close to the Earth and the Moon.

Effective Lagrangian and Effective Potential

1. **[3 pts]** In a polar coordinate system with the center of mass of the heavy masses as the origin, the position of M is $(\nu R, \pi + \Omega t)$ and the position of m is $((1-\nu)R, \Omega t)$ where $\nu = \frac{m}{M+m}$. If the trajectory of the light object in these coordinates is $(r(t), \theta(t))$ show that the Lagrangian for the light object is:

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + G(M+m)\left[\frac{1-\nu}{\rho_1(t)} + \frac{\nu}{\rho_2(t)}\right] \tag{2}$$

where

$$\rho_1(t) = \sqrt{r^2 + \nu^2 R^2 + 2\nu R r \cos[\theta - \Omega t]} \tag{3}$$

and

$$\rho_2(t) = \sqrt{r^2 + (1-\nu)^2 R^2 - 2(1-\nu)Rr\cos[\theta - \Omega t]} \tag{4}$$

2. **[1 pts]** Is the energy of the light object conserved in the coordinate system used in (2)?

3. **[1 pts]** We can go to a rotating frame by making the change of variables $\chi = \theta - \Omega t$. Compute the Lagrangian in the rotating frame.

4. **[2 pts]** Show that in the rotating frame the Hamiltonian is

$$H = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\chi}^2 - V(r,\chi) \tag{5}$$

where

$$V(r,\chi) = -G(M+m)\left[\frac{r^2}{2R^3} + \frac{1-\nu}{r_1} + \frac{\nu}{r_2}\right] \tag{6}$$

Here r_1 is the distance between the light object and M:

$$r_1 = \sqrt{r^2 + \nu^2 R^2 + 2\nu Rr\cos\chi} \tag{7}$$

and r_2 is the distance between the light object and m:

$$r_2 = \sqrt{r^2 + (1-\nu)^2 R^2 - 2(1-\nu)Rr\cos\chi} \tag{8}$$

5. **[2 pts]** What is the physical interpretation of each of the terms in equation (6)?

6. **[2 pts]** Show that up to a constant the effective potential $V(r,\chi)$ can be written in terms of the distances r_1 and r_2:

$$V(r_1,r_2) = -G\left[M\left(\frac{r_1^2}{2R^3} + \frac{1}{r_1}\right) + m\left(\frac{r_2^2}{2R^3} + \frac{1}{r_2}\right)\right] \tag{9}$$

Equilibrium Points

We will now look for special equilibrium positions r_*, χ_* with $0 = \frac{\partial V}{\partial r}(r_*,\chi_*) = \frac{\partial V}{\partial \chi}(r_*,\chi_*)$

7. **[3 pts]** Show that the equilibrium conditions can be cast as a matrix problem

$$0 = \mathbb{M}\cdot\vec{f} \quad \text{where} \qquad \vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \text{and} \qquad f_1 = 1/R^3 - 1/r_1^3\,,\ f_2 = 1/R^3 - 1/r_2^3 \tag{10}$$

Show that two of the entries of the matrix $\mathbb{M}$ are $\mathbb{M}_{11} = rM + \nu RM\cos(\chi)$ and $\mathbb{M}_{21} = -\nu RrM\sin(\chi)$. What are the other two?

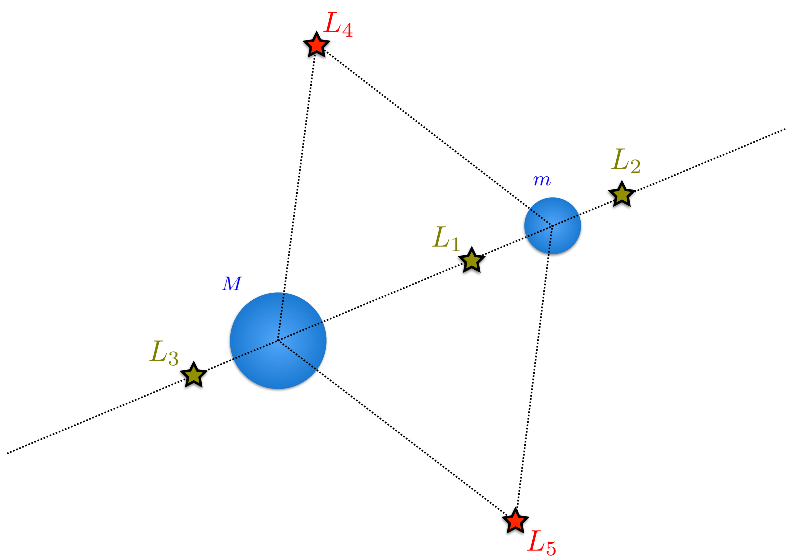


Figure 2: Equilibrium Points.

8. **[2 pts]** One way to satisfy the matrix problem (10) is for the vector to vanish. This yields two of the five critical points L_j depicted in figure 2. Which ones? What are the corresponding values of r_*, χ_*? These points were first found by Lagrange.

9. **[1 pts]** Another possibility is for the determinant of $\mathbb{M}$ to vanish. Show that this does happen when $\sin(\chi)$=0.

10. **[1 pts]** Consider the first the case where $\chi = 0$ and let us look for solutions with r_* bigger than $(1-\nu)R$. The relations between r_1, r_2 and r are now all trivial, without any square roots. Write these relations down.

11. **[3 pts]** Show that in this case the equilibrium condition simply reduces to $h(\rho) = 0$ with $\rho = r_2/R$ and

$$h(\rho) \equiv (1-\nu) + \rho - \frac{1-\nu}{(1+\rho)^2} - \frac{\nu}{\rho^2} \tag{11}$$

Note that $\rho > 0$ by definition and each term in the right hand side has a well defined sign. Sketch the function $h(\rho)$ and explain why there is indeed one (and only one) critical point with $\chi = 0$ and r_* bigger than $(1-\nu)R$. Which point is this in figure 2?

12. **[4 pts]** Establish the existence of the remaining two critical points in figure 2.

Several planets have satellites near their L_4 and L_5 points (which are stable equilibrium points when $\nu < \frac{1}{2}(1-\sqrt{1-4/27})$) with respect to the Sun, with Jupiter in particular having more than a million of these. Artificial satellites have been placed at L_1 and L_2 with respect to the Sun and Earth, and Earth and the Moon, for various purposes, and the various Lagrangian points have been proposed for a variety of future uses in space exploration. [Source: Wikipedia]

3 Biological Population Dynamics

Population dynamics describes how the number of individuals of a given species $N(t)$ changes in time t due to biological processes.[1]

Exponential and logistic growth

The simplest law for the growth of a population is *Malthus law*, which states simply that the *per capita* growth rate of the population is constant:

$$\frac{dN}{dt} = rN\,, \tag{12}$$

where $r > 0$ is the intrinsic growth rate. An obvious problem with this law is that it predicts indefinite exponential growth of a population. A simple improvement which depletes the growth rate at high population numbers is given by the *logistic equation*

$$\frac{dN}{dt} = rN(1 - N/K)\,, \tag{13}$$

where $K > 0$.

1. [**2 pts**] Find the solution $N(t)$ to the logistic equation with initial population $N(0) = N_0$.

2. [**2 pts**] Calculate the value of $N(t)$ when $t \to \infty$. Does it depend on N_0?

Spatial dynamics

The simple example above does not take into account how a population spreads in space, as it implicitly only describes the total population in a certain region. If we want to describe the population dynamics in space and time, we must describe the population by a density, which is a function of space and time. For sake of simplicity, we will take space to be one-dimensional, and the density will be just a function $\rho(x,t)$ so that

$$N(t) = \int dx\, \rho(t,x)\,. \tag{14}$$

The simplest assumption about how a population redistributes itself is that the movement of individuals is akin to a Brownian motion. This leads to the following equation:

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} + r\rho(1 - \rho/K) \tag{15}$$

where $D > 0$ is the diffusion constant and r and K are both positive constants. We further assume that the population is contained in a domain $x \in [0, L]$ and we will impose Dirichlet boundary conditions

$$\rho(0,t) = \rho(L,t) = 0$$

[1] When the number of individuals is sufficiently large, we may describe it by a positive real number varying with time.

3. **[2 pts]** Show that – by conveniently rescaling time, space and the density ρ – the problem above with parameters K, r, L, D is equivalent to

$$\sigma(0,T) = \sigma(1,T) = 0\,, \qquad \frac{\partial\sigma}{\partial T} = A\frac{\partial^2\sigma}{\partial X^2} + \sigma(1-\sigma)\,, \qquad N(t)/K = \int_0^1 dX\,\sigma(X,T) \tag{16}$$

What is A in terms of the original parameters?

4. **[3 pts]** If $\sigma << 1$, we can neglect the quadratic term in the above equation. The resulting linear equation can be solved by separation of variables. Find a solution that obeys the boundary conditions and is everywhere positive, except at the boundary of the domain.

5. **[4 pts]** The solution found above can increase or decrease depending on the domain size. Find a critical value L_c, such that if $L < L_c$ the solution goes to zero for large times.

Predator-prey dynamics

The previous cases refer to *single-species populations*, that is, a population that does not depend of the presence of other populations. It is more realistic to consider the dynamics of interdependent species, said to be *interactive* species. The first model to consider this situation was the Lotka-Volterra model of two species, one being the predator $P(t)$ and the other the prey $N(t)$. In the absence of the predator the prey grows exponentially. When no prey is present the predator dies out. The model is given by the equations:

$$\frac{dN}{dt} = N(a - bP)\,, \qquad \frac{dP}{dt} = P(cN - d)\,, \tag{17}$$

where all constants are positive.

6. **[2 pts]** Give a simple intuitive explanation about why all constants a, b, c, d are indeed expected to the positive.

7. **[2 pts]** Show that by a simple rescaling this problem is equivalent to

$$\frac{du}{d\tau} = u(1 - v)\,, \qquad \frac{dv}{d\tau} = \alpha v(u - 1)\,, \tag{18}$$

with a single dimensionless parameter α.

8. **[4 pts]** Find a conservation law for the system of equations (18). That is, find a function

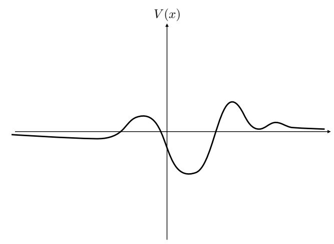
$$F(u, v) = u - \log(u) + \ldots$$

of u and v such that this function is constant in time τ when u and v satisfy equations (18).

9. **[4 pts]** Argue that the Lotka-Volterra dynamics yields a periodic behaviour for the populations of preys and predators.

4 The Inverse Problem in Quantum Mechanics

Consider the one dimensional Schrodinger equation $-\psi''(x) + V(x)\psi(x) = E\,\psi(x)$ for some smooth potential $V(x)$ vanishing at infinity. We can scatter waves from minus to plus infinity and we might support bound-states if the potential is deep enough. Usually we are given a potential from which we compute the scattering data consisting the reflection and transmission coefficients as well as the bound-state data.



In this problem we consider the inverse problem: Given the scattering data, can we fully reconstruct the potential $V(x)$? We encode the scattering data in the function $F(x)$

$$F(x) \equiv \int \frac{dk}{2\pi} R(k) \exp(ikx) + \sum_{n=1}^{N} r_n \exp(-\kappa_n x)\,, \qquad r_n \equiv i\frac{T^2(i\kappa_n)}{T'(i\kappa_n)}\, b_n \tag{19}$$

where N is the number of bound-states that the potential supports and all other quantities are read off from the asymptotic of the scattering states $\psi_k(x)$ (which have continuous positive energy $E = k^2$) and of the bound-states $\psi_n(x)$ (which have discrete negative energy $E = -\kappa_n^2$ with $n = 1, \ldots, N$) as

$$\psi_k(x) \simeq \begin{cases} T(k)\, e^{-ikx} & , \quad x \to -\infty \\ R(k)\, e^{+ikx} + e^{-ikx} & , \quad x \to +\infty \end{cases} \,, \qquad \psi_n(x) \simeq \begin{cases} e^{+\kappa_n x} & , \quad x \to -\infty \\ b_n\, e^{-\kappa_n x} & , \quad x \to +\infty \end{cases} .$$

Given a scattering data $F(x)$ we can reconstruct[2] the potential $V(x)$ as

$$V(x) = -2\frac{d}{dx}K(x,x) \tag{20}$$

where the kernel K is the solution to the Gelfand-Levitan-Marchenko integral equation

$$K(x,y) + F(x+y) + \int_x^\infty dz\, K(x,z)F(z+y) = 0\,. \tag{21}$$

In this problem we explore this very non-trivial equation.

We consider first the case of a reflectionless potential with a single bound-state so that $F(x) = Ae^{-Bx}$.

1. **[2 pts]** Show that $K(x,y) = f(x)g(y)$ where the y dependence is a simple exponential.

2. **[3 pts]** Find $f(x)$ and $V(x)$.

[2]up to simple shifts of x which would not affect the scattering data

3. **[2 pts]** Show that by a simple shift of x we can simplify that potential into $-B^2/\cosh^2(Bx)$.

We see that the potential $V(x) = -B^2/\cosh^2(Bx)$ is very special: It is reflectionless so that $R(k) = 0$ for any $k \in \mathbb{R}$ and it supports a single bound-state with $E = -B^2$.

We will now verify this explicitly. To do so we note that the differential equation $-\psi''(x) - B^2/\cosh^2(Bx)\psi(x) = E\,\psi(x)$ can actually be solved analytically yielding

$$\psi(x) = c_1\, e^{-i\sqrt{E}x}\left(B\tanh(Bx) + i\sqrt{E}\right) + c_2\, e^{i\sqrt{E}x}\left(B\tanh(Bx) - i\sqrt{E}\right)$$

where c_1 and c_2 are two integration constants.

4. **[3 pts]** Consider first bound-states so that the energy is negative and $i\sqrt{E} = -\kappa$. Note that we can take both κ and B to be positive without loss of generality. What should c_1, c_2 and – most importantly – κ be for a normalizable wave function?

5. **[3 pts]** Next we consider scattering states where the energy is positive and $i\sqrt{E} = ik$. By imposing the large x assymptotics of the wave function fix the integration constants in this case as well and show that indeed $R(k) = 0$.

6. **[2 pts]** What is the transmission coefficient $T(k)$ for this simple potential? Where are its poles in the complex k plane?

Finally take a reflectionless potential with N bound-states: $F(x) = \sum_n A_n e^{-\kappa_n x}$.

Inspired by the previous section we make an ansatz $K(x,y) = \sum_n f_n(x) e^{-\kappa_n y}$

7. **[5 pts]** Show that in this case the problem then reduces to a simple linear algebra matrix problem of the form

$$\mathbb{M}\cdot\begin{pmatrix} f_1(x) \\ \vdots \\ f_N(x) \end{pmatrix} = \begin{pmatrix} -A_1 e^{-\kappa_1 x} \\ \vdots \\ -A_N e^{-\kappa_N x} \end{pmatrix} \tag{22}$$

where $\mathbb{M}$ is a simple $N \times N$ matrix. (By inverting this simple problem we can then find f_n and thus $K(x,y)$ and the potential.)

Small potential

Suppose we have a very small potential disturbance such that there are no bound-states and very small reflection coefficient

$$R(k) = \alpha\, f(k) \tag{23}$$

where α can be taken to be very small.

8. **[5 pts]** How would you compute the potential $V(x)$ as a series expansion in α from (21)? Write down the first two terms of the expansion of $V(x)$ in terms of simple integrals of $f(k)$.