Nonlinear phenomena in biology. Exercises.

July 7, 2019

1. **Master equations.** Consider a birth/death process with a Master equation of the form

$$\frac{\partial P(n,t)}{\partial t} = b(n-1)P(n-1,t) + \delta(n+1)^2 P(n+1,t) - (\delta n^2 + bn)P(n,t), \tag{1}$$

where b and δ are the per capita birth and death rate respectively. Obtain the equation for the temporal evolution of $\langle n^2 \rangle$.

2. **Allee effect.** For a population that changes in time according to

$$\dot{N} = rN\left(\frac{N}{A} - 1\right)\left(1 - \frac{N}{K}\right), \tag{2}$$

with r, A and K positive, obtain the stability of its steady states (graphically and analytically) and discuss its biological implications.

3. **Mutualism and predation.** The dynamics of two species that interact with each other in a mutualistic way can be described by,

$$\begin{aligned} \dot{n}_1 &= r_1 n_1 \left(1 - \frac{n_1 + \alpha_{21} n_2}{K_1}\right) \\ \dot{n}_2 &= r_2 n_2 \left(1 - \frac{n_2 + \alpha_{12} n_1}{K_2}\right) \end{aligned} \tag{3}$$

with $\alpha_{21} > 0$ and $\alpha_{12} > 0$. K_1 and K_2 are positive constants. Write the system using nondimensional variables and parameters, obtain its fixed points and their stability. What are the differences between this case and the competitive system studied in the lectures? Consider the case of a predatory interaction, modify the model equations (3) accordingly and discuss the main differences with the mutualistic case studied earlier.

4. **Density-dependent diffusion.** Consider a population that disperses in a $1D$ space with a diffusion coefficient that depends on the local density of individuals

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2}{\partial x^2}\left[D(\rho)\rho\right], \tag{4}$$

with $D(\rho) = D_0(\rho/\rho_0)^p$. ρ_0 is the total density of individuals and $p > 0, D_0 > 0$.

- What is the biological meaning of the choice made for $D(\rho)$?
- Perform a linear stability analysis and discuss the existence of pattern solutions.

5. **Nonlocal density-dependent diffusion.** Consider a diffusion equation of the form

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2}{\partial x^2}\left[D(\tilde{\rho})\rho\right], \tag{5}$$

with $D(\tilde{\rho}) = D_0(\rho/\rho_0)^p$ and $\tilde{\rho} = \int dx' G(|x - x'|)\rho(x, t)$. ρ_0 is the total density of individuals and $p > 0, D_0 > 0$.

- Perform a linear stability analysis for a general kernel function G and discuss the conditions for which patterns can form.
- Discuss the case $p < 0$.