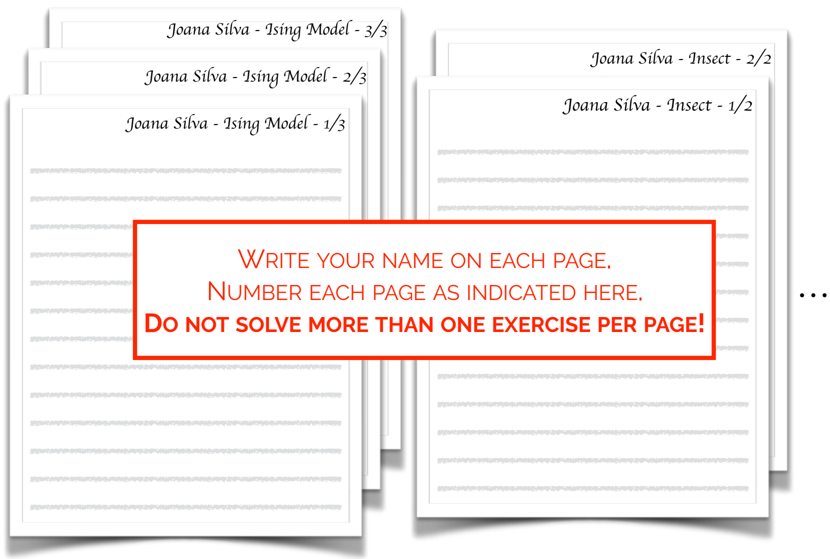
IFT-Perimeter-SAIFR

Journeys into Theoretical Physics 2019

Friday Exam

• Write your name on each page • Number each page as indicated • Do not solve more than one exercise per page.



- Problem 1 (Experimental High-Energy Physics): 25%
- Problem 2 (Ising Model on a Random Lattice): 25%
- Problem 3 (Nonlinear Dynamics of an Insect Outbreak): 25%
- Problem 4 (EFT Methods and Inflation): 25%

○ Full Name: ______________________________

○ I am also interested in the Princeton/CUNY program in biological physics: $\boxed{\text{Yes}}$ $\boxed{\text{No}}$

○ I am interested in applying to the IFT masters program even if I am not accepted into the PSI or Princeton/CUNY program: $\boxed{\text{Yes}}$ $\boxed{\text{No}}$

○ If accepted into any of the programs, I would be interested in starting my fellowship at the IFT in August 2018: $\boxed{\text{Yes}}$ $\boxed{\text{No}}$

○ The areas of physics which I am most interested in are:

Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 Experimental High-Energy Physics

Always assume that $\hbar = c = 1$. So, masses, energies and momenta are all given in GeV.

Reconstruction

1. **[7pt]** Suppose your tracker detector is immersed in a 1T solenoidal field in the z direction. In the (x,y) plane, you obtained the following four hits in meters:

   - $(0, 0)$ • $(1, 0.04)$ • $(4, 0.04)$ • $(5, 0)$

   You can assume that the particle has charge $q = 1e$. Estimate its transverse momentum in GeV. Remember the equation $R = p_{\mathrm{T}}/(0.3 \cdot q \cdot B)$.

2. **[7pt]** Assume that the particle momentum has no z component, i.e., the dip angle λ is zero. Then, the total momentum is just the transverse momentum. The total momentum by itself doesn't give you much information on which particle it was. Luckily, you are able to correlate the track with a signal in the timing detector, and are able to estimate the speed of the particle as $v = 0.99945\,c$.

   Estimate the mass of the particle in GeV. Can you say which particle it is now?

Particle Name	Charge	Mass	Particle Name	Charge	Mass
Tau Lepton	± 1	1.78 GeV	Charged Kaon	± 1	494 MeV
Neutron	0	940 MeV	Charged Pion	± 1	140 MeV
Proton	± 1	938 MeV	Muon	± 1	106 MeV
K-Long	0	498 MeV	Electron	± 1	511 keV

Data Analysis

You are using collision data from the previous accelerator at CERN, the Large Electron Positron (LEP) collider, to search for a new physics particle that couples to leptons, but *not* to quarks. It is electrically neutral, and does not have hadronic interactions of any sort. So, you are searching for reactions of the type:

$$\mathrm{e}^+\mathrm{e}^- \rightarrow X \rightarrow \ell^+\ell^-. \tag{1}$$

For simplicity, consider only decays into electron pairs or muon pairs. Consider, also for simplicity, that the particle momentum lies very close to the transverse plane.

The special thing about this search is that the particle is quasi-stable. Depending on the model parameters, X can fly up to 20 meters before decaying into leptons!

Consider a hermetic detector with cylindrical geometry $\{\rho, \eta, \phi\}$, and 4 subsystems:

- a silicon tracker covering the region $\rho < 1\,\mathrm{m}$,
- an EM calorimeter covering $1\,\mathrm{m} < \rho < 2\,\mathrm{m}$,
- a hadronic calorimeter covering $2\,\mathrm{m} < \rho < 4\,\mathrm{m}$,
- a muon spectrometer covering $4\,\mathrm{m} < \rho < 10\,\mathrm{m}$.

3. **[11pt]** Describe what kind of signal do you expect in each subsystem if the X particle decays after flying for: i) 15 cm ii) 150 cm iii) 15 m.

2 Ising Model on a Random Lattice

Consider the partition function of an Ising spin model on a random lattice where each spin σ_i has four neighboors. That is, the partition function for spins on a dynamical network where we sum over which vertices are connected, how many vertices the network has etc,

$$-\mathbb{F} \equiv \sum_{\text{connected graphs } G_{n,g} \text{ of genus g and } n \text{ vertices}} N^{2-2g} \, e^{-\beta\mu n} \sum_{\{\sigma_i=\pm1\}} e^{-\beta J \sum_{\langle i,j\rangle} \sigma_i\sigma_j - \beta H \sum_i \sigma_i}$$

where i is a vertex of graph $G_{n,g}$ and $\langle i,j\rangle$ indicates vertices i and j which are neighboors in this graph.

1. **[10pt]** What is the physical meaning of β, H, J, μ, N?

Consider now the two matrix model

$$\mathcal{Z} \equiv \int \mathcal{D}A\,\mathcal{D}B \, \exp\left[-\mathrm{Tr}\left(\begin{pmatrix} A & B \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \cdot \begin{pmatrix} A \\ B \end{pmatrix} + g_1 A^4 + g_2 B^4\right)\right]$$

where A and B are $N \times N$ hermitian matrices.

2. **[15pt]** To which values would you fix the kinetic term matrix a_{ij} and the couplings g_1, g_2 so that (the logarithm of) this matrix model partition function computes precisely the random lattice Ising model partition function defined above?

3 Nonlinear Dynamics of an Insect Outbreak

Spruce budworms are serious pests of certain types of trees. The dynamics of a budworm population can be modeled by a modified logistic equation:

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K}\right) - p(N), \tag{2}$$

where $r_B > 0$ is the linear birth rate of the budworms, $K > 0$ is the carrying capacity that is related to the availability of food (foliage) in the host trees and $p(N)$ is a predation term that accounts for the presence of birds feeding on the budworms.

1. **[5pt]** Considering a predation term of the form

   $$p(N) = \frac{AN^2}{B^2 + N^2} \tag{3}$$

   with $A,\ B > 0$, plot $p(N)$ and discuss its behavior. What is the meaning of the parameters A and B?

2. Using the following nondimensional quantities

   $$u = \frac{N}{B}, \quad r = \frac{B\ r_B}{A} \quad q = \frac{K}{B}, \quad \tau = \frac{A\ t}{B},$$

   (a) **[5pt]** Obtain the nondimensional form of Eq. (2) with the specific predation term of Eq. (3).

   (b) **[6pt]** Show, using a graphical analysis, that the model has either three or one nontrivial ($u \neq 0$) steady states depending on the values of q and r. What is the stability of each of the steady states in each case?

3. **[9pt]** Consider a case in which r and q are such that the population growth rate has three nontrivial steady states ($u_1 < u_2 < u_3$) with the stability obtained above. Now, the population of budworms not only grows but also spreads in the form of a traveling wave with wavespeed c in a one-dimensional infinite space. For a traveling wave of budworms that joins a region where $u(-\infty, \tau) = u_3$ with another region where $u(\infty, \tau) = u_1$, the sign of the wavespeed is given by

   $$\text{sign}(c) = \text{sign}\left(\int_{u_1}^{u_3} f(u)du\right) \tag{4}$$

   Discuss the conditions under which the insect wave moves into the u_3 region and thus the insect outbreak is reduced.

4 EFT Methods and Inflation

Conceptual Exercise

1. **[5pt]** If all length scales grow during an expanding FRW universe, do the sizes of atoms or black holes also grow with the overall expansion? Why or why not?

Calculational Exercise

The equation governing the growth of density fluctuations for non-relativistic matter in a spatially flat FRW geometry is

$$\ddot{\delta}_{\mathbf{k}} + 2H\,\dot{\delta}_{\mathbf{k}} + \left(\frac{c_s^2\mathbf{k}^2}{a^2} - 4\pi G\rho_{m0}\right)\delta_{\mathbf{k}} = 0\,, \tag{5}$$

where the density of matter is $\rho_m(\mathbf{x},t) = \rho_{m0}(t)[1+\delta(\mathbf{x},t)]$ so $\delta = \delta\rho_m/\rho_{m0}$ is the fractional fluctuation in the matter density, $\mathbf{k}$ is its Fourier label while $a(t)$ is the scale factor and $H = \dot{a}/a$. The Friedmann equation for the background is

$$H^2 = 8\pi G\rho_0/3\,, \tag{6}$$

where ρ_0 is the total background density of the universe (including radiation and dark energy, in principle, in addition to the matter).

2. **[5pt]** For a matter-dominated universe, $\rho_0 \simeq \rho_{m0} \propto 1/a^3$. What is the time-dependence $a(t)$ predicted by the Friedmann equation?

3. **[5pt]** Show that as $c_s\mathbf{k} \to 0$ eq (5) gives power-law solutions of the form $\delta_0 \propto t^n$. What are the two values of n that solve the equation? For the larger of these two values for n compute the power p in the formula $\delta_0 \propto a^p$.

4. **[5pt]** Consider now the transition between radiation and matter domination, for which $\rho_0 = \rho_{m0} + \rho_{r0}$ and so

$$H^2(a) = \frac{8\pi G\rho_0}{3} = \frac{H_{\rm eq}^2}{2}\left[\left(\frac{a_{\rm eq}}{a}\right)^3 + \left(\frac{a_{\rm eq}}{a}\right)^4\right]\,, \tag{7}$$

where radiation-matter equality occurs when $a = a_{\rm eq}$, at which point $H(a = a_{\rm eq}) = H_{\rm eq}$. Verify that using this new expression for H in (5) implies $\delta_0(x)$ satisfies

$$2x(1+x)\,\delta_0'' + (3x+2)\,\delta_0' - 3\,\delta_0 = 0\,, \tag{8}$$

where the scale factor, $x = a/a_{\rm eq}$, is used as a proxy for time and primes denote differentiation with respect to x. Show that this is solved by a binomial, $\delta_0 \propto \left(x^q + \frac{2}{3}\right)$, and determine the power q. What does this say about the relative speed of growing mode during matter domination as compared with during radiation domination?

5. **[5pt]** Show that the linearly independent solution to the one found in the previous problem only grows logarithmically with x deep in the radiation-dominated era, for which $x \ll 1$.