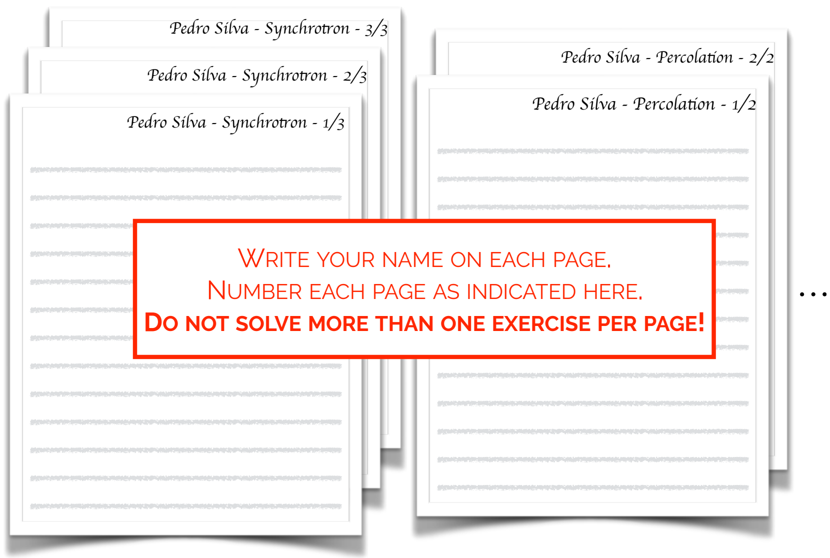
IFT-Perimeter-SAIFR

Journeys into Theoretical Physics 2019

Thursday Exam

• **Write your name on each page • Number each page as indicated • Do not solve more than one exercise per page.**



• Problem 1 (Percolation): 25%
• Problem 2 (Quantum Pigeonholes): 25%
• Problem 3 (Decay of a Relativistic Particle): 25%
• Problem 4 (Synchrotron Radiation: a Simplified Approach): 25%

○ Full Name: ____________________

○ I am also interested in the Princeton/CUNY program in biological physics: Yes No

○ I am interested in applying to the IFT masters program even if I am not accepted into the PSI or Princeton/CUNY program: Yes No

○ If accepted into any of the programs, I would be interested in starting my fellowship at the IFT in August 2018: Yes No

○ The areas of physics which I am most interested in are:

____

Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 Percolation

Consider a lattice where each node can be occupied with probability p or empty with probability $1-p$. A cluster is a set of connected occupied nodes. In the example in figure 3a we have a two dimensional square lattice with 3 clusters; in figure 3b we have the so called Bethe lattice and two clusters; in figure 3c we have a one dimensional lattice with two clusters.

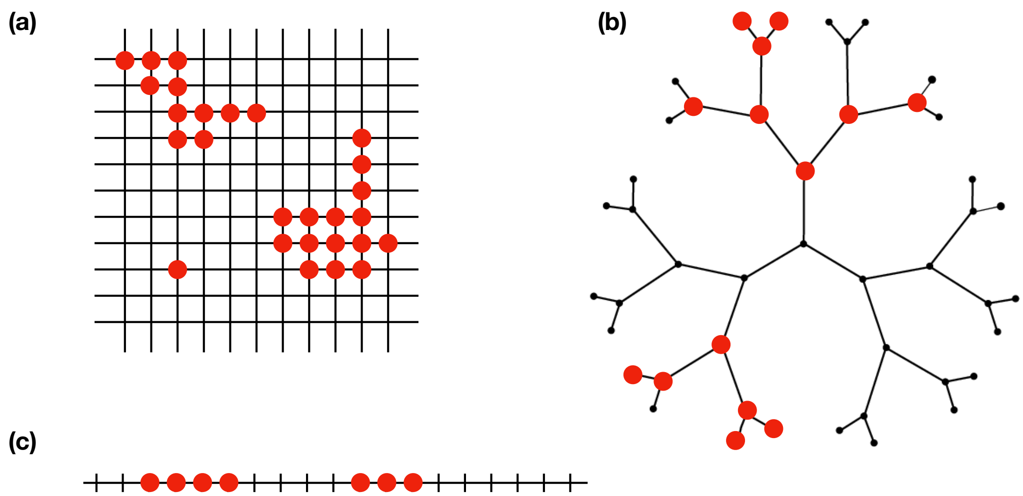


Figure 1: 2D Square lattice, Bethe lattice and 1D lattice with 3, 2 and 2 clusters respectively. The Bethe lattice a connected cycle-free graph where each node is connected to z neighbours, where z is called the coordination number (in the figure $z = 3$). It is a rooted tree, with all other nodes arranged in shells around the root node, also called the origin of the lattice.

As the lattice size goes to infinity we can also have the so called *infinity cluster*, a cluster which occupies a finite fraction of the lattice sites. We define the percolation threshold p_c as the probability p at which an infinite cluster appears for the first time in an infinite lattice. The order parameter in percolation is then the probability P that a site chosen at random will belong to the infinite cluster. A generic expected behavior of P as a function of the occupation probability p is given in figure 2. There is a critical percolation probability p_c below which $P = 0$; as $p \to 1$ the order parameter goes to 1 since all sites are occupied. For $p \to p_c^+$ slightly above the critical value we expect

$$P \sim (p - p_c)^\beta \tag{1}$$

where β is a critical exponent.

Square Lattice

Consider here an infinite square lattice. We use $n_s(p)$ to denote the number of clusters of size s per lattice site.

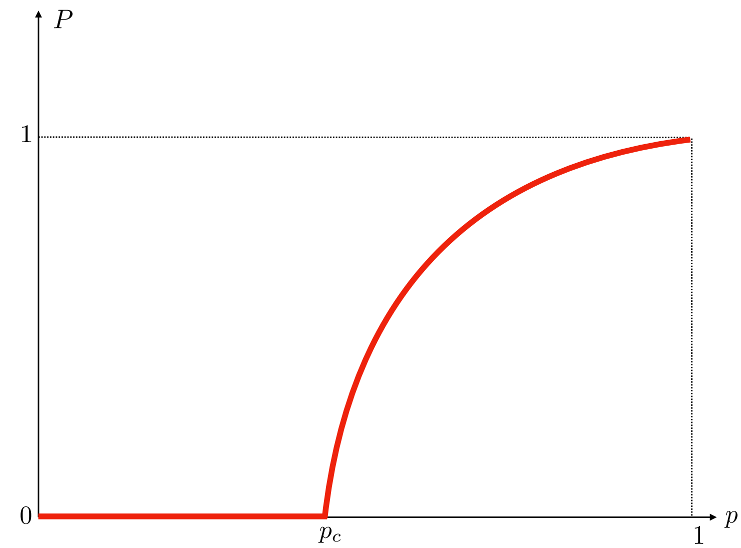


Figure 2: Probability P of a random site becoming to an infinite cluster as function of the single site occupation probability p.

1. **[3pt]** Show that the probability $n_1(p)$ of finding a single isolated occupied site (i.e. a cluster of size 1) in a square lattice is

$$n_1(p) = p(1-p)^4 \tag{2}$$

2. **[3pt]** What is the probability $n_2(p)$ of finding a cluster of two sites in this lattice?

   Hint: If $p = 1/2$ you should get $n_2(1/2) = 1/2^6$.

Percolation in 1D

In 1D, the lattice is a simple infinite line and everything can be solved analytically.

3. **[3pt]** Show that

$$n_s(p) = (1-p)^A p^B \tag{3}$$

   and find A and B.

4. **[3pt]** Show that $P = 0$ for any $p < 1$. Explain why this is the expected result.

Percolation in a Bethe Lattice

Next we consider a Bethe lattice with coordination number $z = 3$ (i.e. each site has three neighboors) as in the example in figure 3b. We use g to indicate he number of generations of the lattice, that is the distance of the most outer noted to the centre node. In figure 3b we have $g = 4$ for example; in that figure we have a total number of 46 sites out of which 24 are in the outermost layer and are thus called *surface sites*.

5. **[3pt]** Show that, for a general Bethe lattice of coordination z,

$$\frac{\text{Number of Surface sites}}{\text{Total Number of Sites}} \to \frac{z-2}{z-1} \tag{4}$$

as $g \to \infty$. What is interesting here is that the number of surface sites approaches a finite fraction of the total number of sites as the lattice size is sent to infinity.

6. **[5pt]** Assume an infinite lattice henceforth. Define Q as the probability that an arbitrary site is *not* connected to infinity through a fixed branch originating at this site. Argue that Q obeys the nice relation

$$Q = (1-p) + p\, Q^{z-1} \tag{5}$$

and use it to find Q in the example in the figure where $z = 3$.

Hint: The solution has two branches depending on wheather $p < p_c$ or $p > p_c$ where p_c takes a simple value which you should compute.

7. **[5pt]** Argue that

$$P = p\,(1 - Q^z)\,. \tag{6}$$

What is P for the $z = 3$ case and what is the critical exponent β defined in (1)?

2 Quantum Pigeonholes (with Rabbits)

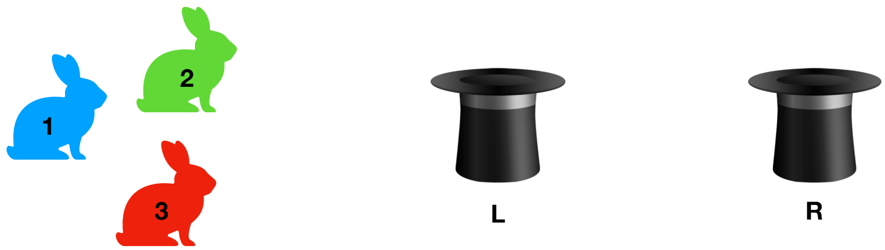


Figure 3: 3 distinguishable rabbits (or particles) are put in two boxes. Quantum mechanically, we can get very counterintuitive results for the probabilities of encountering various combinations of particles in various combinations of these boxes.

The Classical Problem

Consider three distinguishable rabbits labeled by $1, 2, 3$ which can be put into two boxes labeled L and R. Someone hides the rabbits in these boxes in a random way. Let P_{ij} denote the probability of having rabbits i and j both in the same box (so that $1 - P_{ij}$ is the probability of having rabbits i and j in different boxes).

1. **[1pt]** If $P_{12} = P_{23} = 0$, what is P_{13}?
2. **[1pt]** If $P_{12} = P_{23} = 1$, what is P_{13}?
3. **[1pt]** If $P_{12} = 0$ and $P_{23} = 1$, what is P_{13}?

The Quantum Problem

Here we consider now three distinguishable quantum particles labelled by $1, 2, 3$ which can again be put into two boxed L and R. Particle 2 in the R box would be represented by the ket $|R\rangle_2$ for example.

4. **[2pt]** What is the projection operator Π_{12} into the subspace where particles 1 and 2 occupy the same box?
5. **[3pt]** We initially prepare the system in the state

$$|\Psi\rangle = |+\rangle_1|+\rangle_2|+\rangle_3\,, \qquad |+\rangle_a \equiv \frac{|L\rangle_a + |R\rangle_a}{\sqrt{2}}\,, \tag{7}$$

What is the probability that particles 1 and 2 occupy the same box? What about 2 and 3? What about 1 and 3? What classical random process would generate these same probabilities?

6. **[2pt]** Next we consider a measurement to find out if each of the particle is in the state $|+i\rangle = \frac{1}{\sqrt{2}}(|L\rangle + i|R\rangle)$ or the state $|-i\rangle = \frac{1}{\sqrt{2}}(|L\rangle - i|R\rangle)$. What is the probability of finding the possibility

$$|\Phi\rangle = |+i\rangle_1 |+i\rangle_2 |+i\rangle_3 \tag{8}$$

if we start with the state (7)?

7. **[2pt]** In a different setup, we want to do something else before measuring the state in the basis $|\pm i\rangle_1 |\pm i\rangle_2 |\pm i\rangle_3$. We would like to perform an intermediate measurement to determine if particle 1 and particle 2 are in the same box or not. Note that we do not wish to know in which exact box they are, just the relative location. What state does the state (7) collapse into, at this intermediate step, if the particles are indeed found to be in the same box?

8. **[2pt]** Assuming we collapse into that state, what is the probability of measuring the state $|\Phi\rangle$ in the final state?

9. **[2pt]** What would happen if you were to repeat this exercise for other pairs of particles?

10. **[3pt]** How does this compare with the classical intuition of the first problem of the first part?

11. **[6pt]** Repeat points 6 to 9 with $|\Phi\rangle$ replaced by $|\Phi'\rangle = |+i\rangle_1 |-i\rangle_2 |+i\rangle_3$. How do your findings compare with the classical intuition of the second problem of the first part?

3 Decay of a Relativistic Particle

A particle A of mass M has a half-life (at rest) of 10 seconds to decay into two stable particles B and C which both have mass $\frac{M}{4}$.

Suppose particle A is moving at $v = \frac{3}{5}c$.

1. **[2pt]** What is its half-life?

When particle A decays into B and C, suppose that particle B emerges at an angle perpendicular to the direction of particle A.

2. **[3pt]** What is the magnitude of the velocity of particles B and C in the center-of-mass reference frame?

3. **[6pt]** What is the magnitude of the velocity of particles B and C in the original reference frame?

Suppose that particles B and C both have mass $\widetilde{M}$ (instead of $\frac{M}{4}$).

4. **[4pt]** What is the maximum value of $\widetilde{M}$ such that particle B can emerge at an angle perpendicular to the direction of particle A?

5. **[10pt]** If $\widetilde{M}$ is larger than this maximum value, what is the range of possible angles (as a function of $\widetilde{M}$) for particle B to emerge with respect to the direction of particle A?

4 Synchroton radiation: a simplified approach

Accelerated charged particles emit radiation. When this acceleration is perpendicular to the velocity of the particle the emission is called synchroton radiation. This type of radiation has many applications in the study of materials and Brazil is now finishing the construction of one of the most advanced synchroton source in the world, called Sirius (after the brightest star in the sky). The properties of synchroton radiation can be derived using retarded potential in Maxwell's equations (as in Jackson chapter 14) but in this problem we will study it using a simpler method originally developed by J. J. Thomson. This method is not rigorous but I hope it will help to develop some intuition. In fact it is valid for any type of radiation produced by accelerated charges. This problem might look a bit long but there is a lot of text trying to make it easier.

Throughout we use MKS units where the electric field of a static particle with charge q is given by:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r} \tag{9}$$

Consider an electric charge at rest at time $t = 0$. The charge is then accelerated by changing its velocity by Δv in a short time interval Δt. The electric field configuration at a finite time t will look like figure 4, where we are neglecting the motion with constant speed. The field outside the sphere with radius $r = ct$ has to be the same as before the since there was no time for the information about the motion of the charge to reach this region.

The variation of the electric field in the time interval Δt represented in the thin shell in the figure will produce the synchroton radiation.

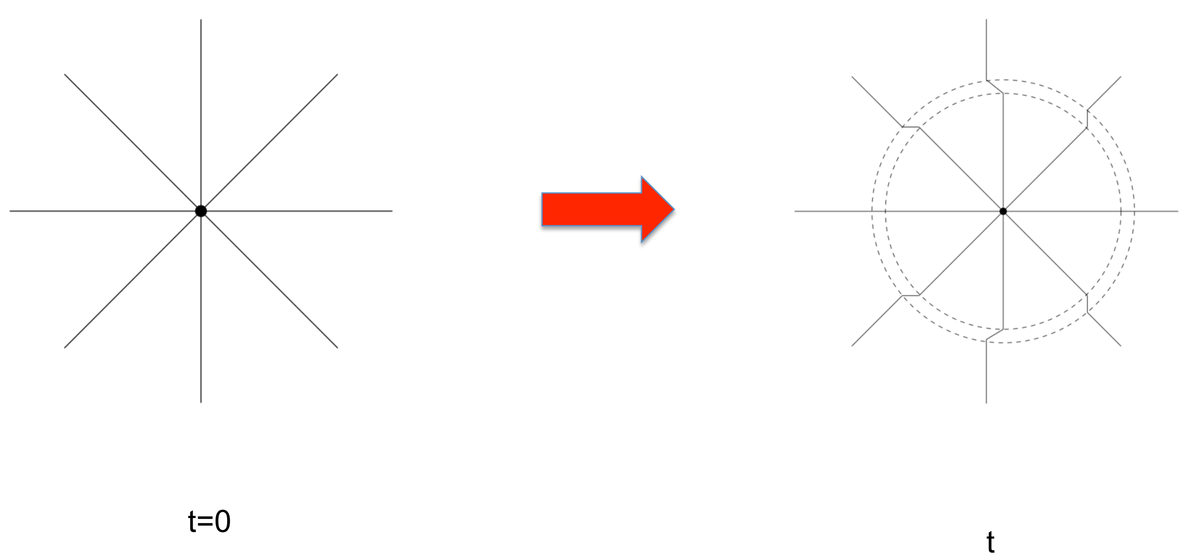


Figure 4: Electric field of the charge before and after the acceleration period of Δt.

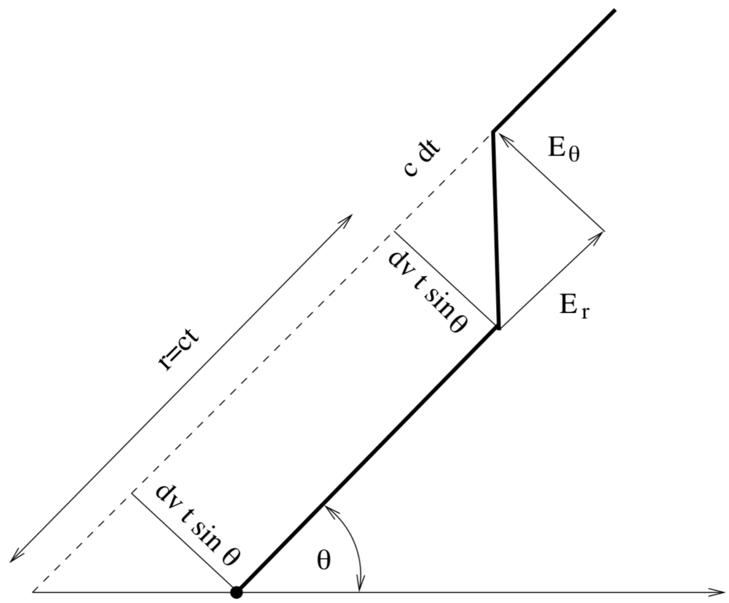


Figure 5: Detailed version of the change in one particular field line. The particle moves to the right, along the x axis. (In the figure $dt = \Delta t$ and same for the velocity.)

Polar electric field

1. **[4pt]** Consider the more detailed figure 5. Show that the electric field in the θ direction can be written as

$$E_\theta = \frac{q}{4\pi\epsilon_0}\frac{1}{rc^2}\dot{v}\sin\theta \tag{10}$$

where $\dot{v} = \Delta v/\Delta t$.

Notice that this polar field decreases slower than the radial field and hence is dominant at large distances. This is the field responsible for the radiation.

Flux of energy: the Poynting vector

The emission of radiation takes out energy from the system. This is quantified by the Poynting vector $\vec{S}$, which is derived from conservation of energy and is given by:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \tag{11}$$

The electric and magnetic fields of a monochromatic electromagnetic plane wave propagating in the $\hat{z}$ direction can be written as:

$$\vec{E} = E_0 e^{i(kz-\omega t)}\hat{x} \tag{12}$$
$$\vec{B} = B_0 e^{i(kz-\omega t)}\hat{y} \tag{13}$$

with $c = \omega/k = 1/\sqrt{\epsilon_0\mu_0}$.

2. **[3pt]** Show that Maxwell's equation in vacuum imply that

$$|\vec{B}| = \frac{1}{c}|\vec{E}| \tag{14}$$

3. **[2pt]** Show that the Poynting vector is given by

$$|\vec{S}| = c\epsilon_0|\vec{E}|^2 \tag{15}$$

Power radiated

The power radiated is given by

$$P = \int \vec{S} \cdot d\vec{A} \tag{16}$$

4. **[3pt]** Show that for the case of E_θ the power emitted is

$$P = \frac{1}{6\pi}\frac{q^2\dot{v}^2}{c^3\epsilon_0} \tag{17}$$

This is known as the Larmor's formula for the energy loss of an accelerated charge.

5. **[2pt]** Show that it can also be written as

$$P = \frac{1}{6\pi}\frac{q^2}{c^3\epsilon_0 m^2}\frac{d\vec{p}}{dt} \cdot \frac{d\vec{p}}{dt} \tag{18}$$

where $\vec{p}$ is the momentum and m is the mass of the charged particle.

Relativistic formula

Now we are ready to guess the relativistic formula for synchroton power emission. We can write a covariant form of Larmor's formula using the 4-momentum $p_\mu = (E/c, \vec{p})$ and the proper time τ. A covariant formula for the emitted power is:

$$P_{rel} = \frac{1}{6\pi}\frac{q^2}{c^3\epsilon_0 m^2}\left|\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right| \tag{19}$$

6. **[2pt]** Show that this equation can be written as

$$P_{rel} = \frac{1}{6\pi}\frac{q^2}{c^3\epsilon_0 m^2}\gamma^2\left|\frac{d\vec{p}}{dt} \cdot \frac{d\vec{p}}{dt} - \frac{v^2}{c^2}(\frac{dp}{dt})^2\right| \tag{20}$$

where $p \equiv \sqrt{\vec{p} \cdot \vec{p}}$ and $\gamma = E/(mc^2)$.

Synchrotron radiation: circular accelerators

Although synchroton radiation is very useful in some cases it is undesirable. This happens when one tries to accelerate particles to the highest possible energies, as in the Large Hadron Collider at CERN in order to study how elementary particles interact. We are now going to put some numbers in our calculations in order to estimate the effect of synchroton radiation in circular accelerators (sorry, I'm a phenomenologist).

7. **[2pt]** Consider a charged particle with fixed energy E confined in a circular orbit of radius R with a magnetic field. In this case the magnitude of the velocity is constant but not its direction. Show that in this case

$$P_{rel} = \frac{1}{6\pi}\frac{q^2}{c^3\epsilon_0}\gamma^4 a^2 \tag{21}$$

where $a = v^2/R$ is the centripetal acceleration.

8. **[2pt]** Finally, consider relativistic particles and use $\gamma = E/(mc^2)$ and use the definition of the fine structure constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0 c\hbar} \simeq 1/137 \tag{22}$$

to get

$$P_{rel} = \frac{2\alpha}{3}\frac{E^4}{(mc^2)^4}\frac{\hbar c^2}{R^2} \tag{23}$$

9. **[3pt]** Now we have to get our units straight. One of my favorites numerics (very handy in particle physics for conversion of units) is:

$$\hbar c \approx 200 \text{ MeV x fermi} \tag{24}$$

where 1 fermi $= 10^{-15}m$. For an electron with mass $mc^2 = 0.5$ MeV and a reference energy of $E = 1GeV$ and reference radius $R = 1$ km show that

$$P_{rel} \approx 5\frac{(E/1 \text{ GeV})^4}{(R/1 \text{ km})^2} \text{ MeV/s} \tag{25}$$

10. **[1pt]** Is this a problem for accelerators? The Large Electron Positron Collider (LEP), that worked in the same tunnel as the LHC in the 1990's ($R \approx 5$ km) accelerated electrons at energies of $E = 160$ GeV. Estimate the energy loss due to synchroton radiation in one revolution at the LEP.

11. **[1pt]** There is right now a discussion whether the next electron-positron accelerator should be linear or circular. CERN has a plan for a future circular collider with energy 2 times larger than LEP and a tunnel which is 3 times larger than LEP. What is the increase in energy loss compared to LEP?