$\hat{M}.\hat{A}.\hat{T}.\hat{R}.1.\hat{X}$

M.O.D.E.L.S

Journeys 19

Outline

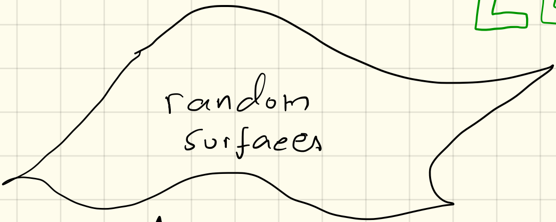
1. Simplest graphs / Gaussian Integrals
2. Gaussian Integration on (Feynman) graphs
3. Graps and maps $\longrightarrow$ topology
4. Matrix models and 2D quantum gravity

References:

SEE LAST PAGE

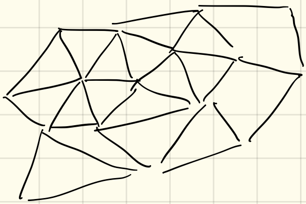
LECTURE 1

Outline



e.g some membrane OR 2D quantum gravity
OR string theory worldsheet
OR ...

see computer games



poligonalization

gaussian integration

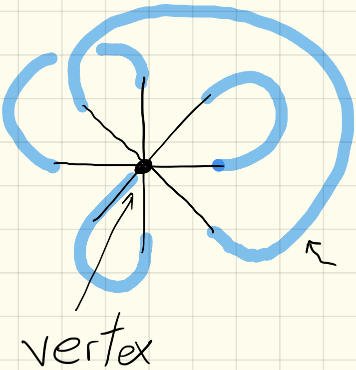
$= \sqrt{2\pi}$

graph 2.0

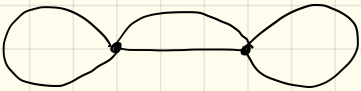
graph counting and map counting

matrix models M_{ij}

Graphs & Gaussian Integrals

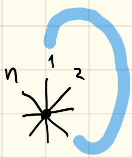


← Example of a graph with valency $z = 8$ and one vertex and four edges



graph with 2 vertices and 4 edges ↑ each of valency 4

1] Exercise: A graph has n vertices, each of valency z. How many edges does it have?



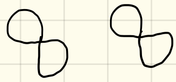
$n-1$ options

$n-3$ options next...

$$\# \text{Ways} = (n-1)(n-3)\dots 3 \cdot 1 = (n-1)!!$$

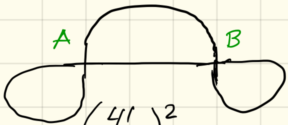
✱ 8 $\longleftrightarrow$ $7!! = 7.5.3 = \boxed{105}$

+ + 2 vertices :



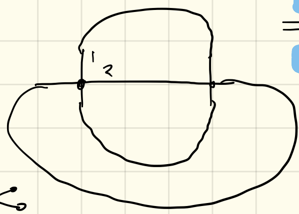
$3 \times 3 = \boxed{9}$

$$\left[\begin{array}{l}\text{symbolic representation:}\\ (A \longleftrightarrow B)^2 \times \\ \times (A \longleftrightarrow A) \times (B \longleftrightarrow B)\end{array} \cong \right]$$



$$\left(\frac{4!}{2!\,2!}\right)^2 \times 2 = \frac{4\times3\times2\times\cancel{4}\times3\times\cancel{2}\times\cancel{2}}{\cancel{2}\,\cancel{2}\;\cancel{2}\,\cancel{4}} = \boxed{72}$$

Who to self-connect in each vertex
× 2 for vs
between vertices



$4 \times 3 \times 2 = \boxed{24}$

where 1 goes to   where 2 goes to, ...

$$9 + 72 + 24 = 105$$

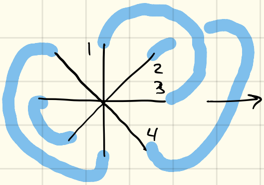
obvious: we could decide 4 of 8 vertices are "special" and separate ✱ into + +

Indistinguishible "particles" factors:

↑ edge = "particles holding hands"

Automorphism factor or symmetry factor:

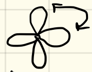
$$\frac{105}{8!} = \frac{1}{8!!} = \frac{1}{8\times6\times4\times2} = \underbrace{\frac{1}{4!\,2^4}}_{\text{automorphism of the graph}} = \frac{1}{\Gamma(\circledast)}$$



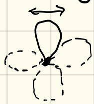
representative (all o.k. same!)

$$\underbrace{(13)(24)(58)(67)}$$

4! from re-ordering pairs

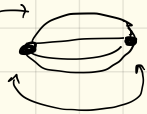


2 for each pair $= 2^4$



$$\frac{9}{(4!)^2\,2!} = \frac{1}{(4\times2)^2\,2} = \frac{1}{\underbrace{2!}_{\text{indistinguishable vertices}}\,2^3\,2^3} \qquad \underbrace{8}_{\frac{1}{\Gamma}=\frac{1}{2^3}}\ 8 = \frac{1}{\Gamma(88)}$$

$$\frac{24}{4!^2\,2!} = \frac{1}{4!\,2!} = \frac{1}{\text{permutations of lines and vertices}}$$



$$Z \equiv \sum_{\substack{\text{graphs of}\\ \text{valency 4}\\ \text{and } n \text{ vertices}}} \frac{g^n}{\Gamma(G)} = 1 + 8 + g^2\left(88 + \infty\!\circ + \ominus\right) + \dots \equiv e^{-F}$$

$1/\Gamma$ implicit

partition function

free energy

$$\underline{\underline{\text{Claim}}} \quad -F = \sum_{\substack{\text{connected}\\ \text{graphs}}} \frac{g^n}{\Gamma(G)}$$

Lets check to $O(g^2)$: $\quad e^{-F} = \underbrace{1 + (-F)}_{} + \frac{(-F)^2}{2} + \dots$

$$= \text{same as } Z \text{ without } 8\ 8 + \frac{1}{2!} 8^2 + \dots$$

$$\underbrace{\qquad\qquad}_{Z} \quad :\text{-D}$$

$\boxed{\begin{array}{l}\text{2] Exercise: check up to } O(g^3) \\ \text{3] Exercice: prove general statement}\end{array}}$

look like a macrocanonical ensemble

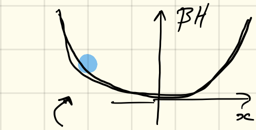
Z is a statistical mech sum of configurations, graphs. These graphs are called Feynman graphs.

Consider now another partition function where we consider a single degree of freedom with

$$\beta H(x) \equiv \frac{x^2}{2} - g\, x^4/4!$$

$$\mathcal{Z} \equiv \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}}\, e^{\overbrace{-\frac{x^2}{2} + g\frac{x^4}{4!}}^{-\beta H(x)}}$$

normalization we will understand soon



$(g<0$ for now$)$

Here we have a single dof in an anharmonic potential.

<u>CLAIM</u>

$$\boxed{\mathcal{Z} = \mathbb{Z}\ .\ \blacksquare}$$

$$\mathcal{Z}\Big|_{g=0} \overset{?}{=} 1$$ is a first obvious check!

Indeed $$\left(\mathcal{Z}\Big|_{g=0}\right)^2 = \frac{1}{2\pi}\int dx \int dy\, e^{-\frac{x^2+y^2}{2}}$$

$$= \overbrace{\frac{1}{2\pi}\int_0^{2\pi} d\theta}^{1} \underbrace{\int_0^{\infty} r\,dr\, e^{-r^2/2}}_{= -e^{-r^2/2}\big|_0^{\infty} = 0-(-1) = +1} = 1 \ \blacksquare$$

$$\boxed{4] \text{ Exercise: show that } \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2/2} = \sqrt{\frac{2\pi}{\alpha}} \equiv I_\alpha}$$

$$\int dx\, x^K e^{-\alpha x^2/2} = \begin{cases} 0 \,, & K \text{ odd} \\ (-2)^n \left(\frac{\partial}{\partial \alpha}\right)^n I_\alpha \,, & K = 2n \text{ even} \end{cases}$$

$$K=2 : \quad (-2)\frac{\partial}{\partial \alpha} \sqrt{2\pi}\, \alpha^{-1/2} = \sqrt{2\pi}\, \alpha^{-3/2}$$

$$K=4 : \quad (-2)\, \partial/\partial\alpha \sqrt{2\pi}\, \alpha^{-3/2} = \sqrt{2\pi}\; 3\, \alpha^{-5/2}$$

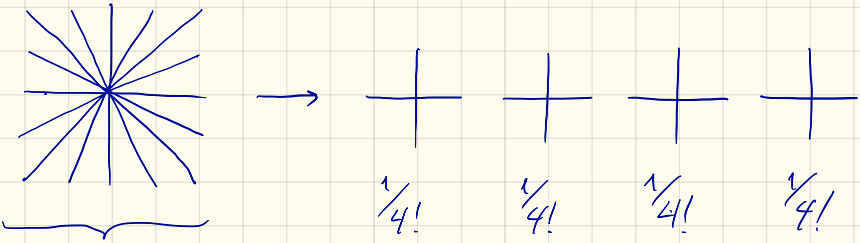
$$K=6 : \quad (-2)\, \frac{\partial}{\partial\alpha} \cdots \alpha^{-5/2} = \sqrt{2\pi}\; 5.3.1\, \alpha^{-7/2} \quad \text{etc}$$

$$\langle x^{2n} \rangle = \int dx\, x^K e^{-\alpha x^2/2} \Big/ \int dx\, e^{-\alpha x^2/2}$$

$$= \underbrace{(2n-1)!!}_{\text{familiar?}} \; \underbrace{\alpha^{-n}}_{\text{dim analysis}}$$

$$Z = \sum_{n=0}^{\infty} \boxed{\frac{g^n}{n!}\left(\frac{1}{4!}\right)^n} \underbrace{(4n-1)!!}_{\#\, * \text{ with } 4n \text{ valency}}$$

splits $4n$ vertex into n vertices of valency 4 as we saw before



$\times \frac{1}{n!}$

number of vertices = 4 in this case

4n valency with $n=4$ in this example

$$\left(\frac{1}{4!}\right)^n \frac{1}{n!} \neq \frac{1}{(4n)!}$$ hence

So indeed $$Z = \sum_{\substack{\text{graphs of valency 4} \\ \text{and } n \text{ vertices}}} \frac{g^n}{\Gamma(G)} \neq \sum_{\substack{\text{graphs with a single} \\ \text{vertex of valency } 4n}} \frac{g^n}{\Gamma(G)}$$

5] Exercise: what is the graph expansion of

$$Z_k \equiv \int_{-\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2} + g\frac{x^k}{k!}}$$

where $k = 3, 4, 5, \ldots$

this exercise is IMPORTANT

6] Exercise: what integral computes the red expression:
$$\sum_{\substack{\text{graphs with}\\ \text{a single vertex of}\\ \text{valency } 4n}} \frac{g^n}{\Gamma(G)} \;?$$

this exercise is a curiosity, NOT important

Sources and a rederivation

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi/\alpha}}\, e^{-\alpha \frac{x^2}{2} + jx} \equiv Z[j]$$

note that we normalized so that $Z[0] = 1$

$$\left.\frac{\partial^n}{\partial j^n} Z[j]\right|_{j=0} = \langle x^n \rangle$$

so $Z[j]$ is a generating function, very useful!

$$Z[j] = \underbrace{\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi/\alpha}} e^{-\frac{\alpha}{2}\left(x - j/\alpha\right)^2}}_{1} \; e^{+\; j^2/2\alpha}$$

$$\boxed{Z[j] = e^{j^2/2\alpha}} \Rightarrow \langle x^{2n+1} \rangle = 0$$

and
$$\langle x^{2n} \rangle = \left(\frac{\partial}{\partial j}\right)^{2n} \left[\ldots + \frac{j^{2n}}{(2\alpha)^n n!} + \ldots \right]_{j=0}$$

too few j too many j

$$= \alpha^{-n} \frac{(2n)!}{\underbrace{2^n n!}_{(2n-1)!!}}$$

$\leftarrow$ all = even × odd

$\leftarrow$ even

$\leftarrow$ odd

also clear combinatorically directly

order does not matter

$$\underbrace{\frac{1}{2^n n!}}\left(\frac{\partial^2}{\partial j^2}\right)^n \underbrace{j \ldots j}_{n}$$

connect two j's

LECTURE 2

Multidimensional Integrals

$$Z[\vec{j}] \equiv \int dx_1 \ldots dx_N \; e^{-\sum_{ij}^{N} x_i A_{ij} x_j / 2 + \sum_{k}^{N} x_k j_k}$$

$$= \int d\vec{x} \; e^{-x^T \cdot A \cdot x/2 + x \cdot j}$$

vector

matrix (symmetric)

(Previous example: $N = 1$, $A_{11} = \alpha$)

Z is again an important partition function since

$$\boxed{\langle x_{i_1} \cdots x_{i_n} \rangle_{\text{Gaussian}} = \frac{1}{Z[0]} \frac{\partial}{\partial j_{i_1}} \cdots \frac{\partial}{\partial j_{i_n}} Z[\vec{j}] \Big|_{\vec{j}=0}}$$

next: $Z[0] = ?$

$Z[\vec{j}] = Z[0] \times ?$

for we actually only need so lets start there.

$$-x_i A_{ik} x_k/2 + x_k j_k$$ (Sum over repeated indices here)

$$\parallel$$

$$-(x_i + B_i) A_{ik} (x_k + B_k)/2 + B_i A_{ik} B_k$$

if $B_k = -A^{-1}_{kl} j_l$ or just $B = -A^{-1} \cdot j$

indeed,

$$-(x - A^{-1} j)^T \cdot A \cdot (x - A^{-1} j)/2$$

use $A = A^T$

$$= -x^T \cdot A \cdot x/2 \quad \left.\begin{array}{l} + x^T \cdot A \cdot A^{-1} \cdot j/2 \\ + j^T (A^{-1})^T \cdot A \cdot x/2 \end{array}\right\} x^T \cdot j \text{ or } \vec{x} \cdot \vec{j}$$

$$- j^T \underbrace{(A^{-1})^T \cdot A \cdot A^{-1}}_{A^{-1}} \cdot j/2$$

$$\boxed{Z[j] = \left[\int dy_1 \ldots dy_N \, e^{-y^T \cdot A \cdot y} = Z[0]\right] e^{\frac{1}{2} j^T A^{-1} j}}$$

$y_k = x_k + B_k$

(Previous example: $\vec{j} = j \in \mathbb{R}$, $A^{-1} = 1/\alpha$)

Conclusion:

$$\boxed{\langle x_{i_1} \cdots x_{i_k} \rangle = \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} \, e^{\frac{1}{2} j \cdot A^{-1} \cdot j}}$$

Wick Theorem:

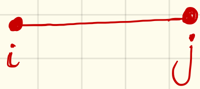
first note that

$$\langle x_i \, x_j \rangle = (A^{-1})_{ij} \equiv \text{propagator of } x$$

obviously $\neq 1/A_{ij}$; really need inverse!

$$\equiv$$

$$\langle \underset{i}{\bullet} \qquad \underset{j}{\bullet} \rangle \equiv$$



cartoon only, use them if you like only, not mandatory of course!

7] Exercise:

Explain $\underbrace{}_{\text{in more than one way}}$ why $\langle X_{i_1} \cdots X_{i_{2n+1}} \rangle = 0$

$$\langle X_{i_1} \cdots X_{i_{2n}} \rangle = \frac{\partial}{\partial j_{i_1}} \cdots \frac{\partial}{\partial j_{i_{2n}}} \left[\text{too few } j\text{'s} + \text{good} + \text{too many } j\text{'s} \right]_{j=0}$$

Where

$$\text{good} = \frac{1}{2^n n!} j_{k_1} j_{k_2} \cdots j_{k_{2n}} \underbrace{(A^{-1})_{k_1 k_2} (A^{-1})_{k_3 k_4} \cdots}_{\langle X_{k_1} X_{k_2} \rangle \langle X_{k_3} X_{k_4} \rangle \cdots}$$

Example:

$$\langle X_{i_1} \cdots X_{i_{2n}} \rangle = \underbrace{\sum_{\text{Wick contractions}}}_{\sum_{n \text{ pairings}}} X_{i_1} X_{i_2} X_{i_3} X_{i_4} \cdots X_{i_{2n-1}} X_{i_{2n}}$$

the example $=$

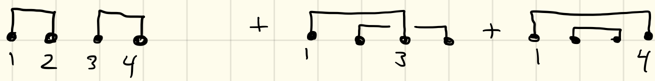
$$\langle X_{i_1} X_{i_4} \rangle \langle X_{i_2} X_{i_3} \rangle \cdots \langle X_{i_{2n-1}} X_{i_{2n}} \rangle$$

$$\boxed{\langle X_{i_1} \cdots X_{i_{2n}} \rangle = \sum_{\text{pairs}} \prod_{a=1}^{n} \langle X_{j_a} X_{k_a} \rangle}$$

$$\{i_1 \cdots i_{2n}\} \to (j_1 k_1)(j_2 k_2)\cdots(j_n k_n)$$

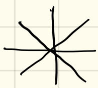
($j_a < j_{a+1}$ and $j_a < k_a$ to avoid over counting)

$$\langle X_{i_1} \cdots X_{i_4} \rangle = \langle X_{i_1} X_{i_2} \rangle \langle X_{i_3} X_{i_4} \rangle +$$

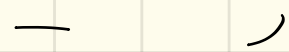


for $N = 1$ trivial case

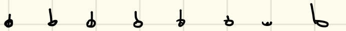
$$\langle \underset{1}{x}\ \underset{2}{x} \cdots \underset{2n}{x} \rangle =$$



$$= (2n-1)!!$$



number of wick contractions



+ ....

So all previous graph statements are now trivial as we construct graphs by adding edges $\equiv$ wick contraction

$$\boxed{Z[0]} = \int d\vec{x}\, e^{-\frac{1}{2}\vec{x}\cdot A\cdot\vec{x}}$$

$\vec{x} = \Lambda y$

diagonalizes A

$$= \int d\vec{y}\, e^{-\frac{1}{2}\vec{y}\cdot\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{pmatrix}\vec{y}}$$

$$= \prod_{j=1}^{N}\sqrt{\frac{2\pi}{a_j}} = \underbrace{\boxed{(2\pi)^{N/2}\Big/\sqrt{\det A}}}_{\text{just normalization for most of these lectures}}$$

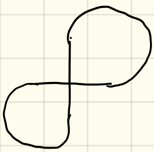
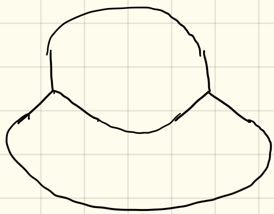
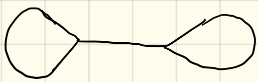
* A ← action
* $\det A$ ← zero source partition fn
* A^{-1} ← propagator and correlators

(in previous example $\det A = A_{11} = \alpha$)

Lema

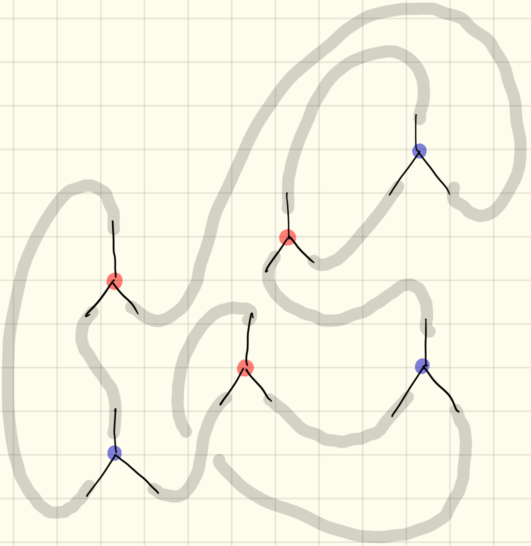
$$\int dx \, e^{-\frac{x^2}{2} + \sum_{k \geq 3} g_k x^k / k!}$$

$$= \sum_{\substack{\text{graphs } G \\ \text{with } n_A \\ \text{vertices} \\ \text{of valency } A}} \prod_{k=3}^{\infty} (g_k)^{n_k} \frac{1}{\Gamma(G)}$$

$= 1 + g_4$  $+ g_3^2 \Big[$  $+$  $\Big]$

$+ \ldots$

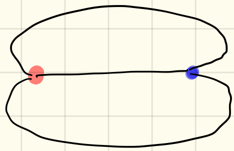
8] Exercise: Is there a term proportional to $g_4 g_3$? What about $g_4 g_3^2$?
Describe generically which terms show up.



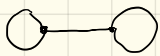
$\equiv \mathcal{Z}$

with red and blue vertex where blue connects to red only and all vertices are cubic

$\parallel\!\!|$

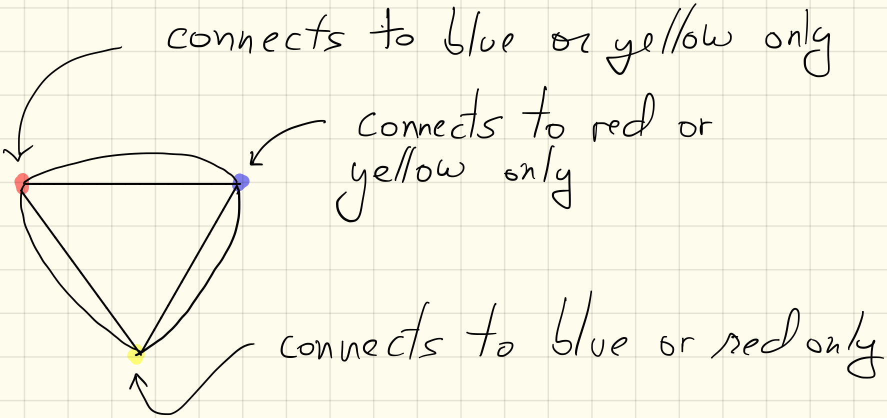
$$\sum \frac{g^{2n}}{\Gamma(G)} = 1 + $$  $+ \dots$

red vertices
$\shortparallel$
blue vertices
$\shortparallel$
$n \geqslant 0$

( forbidden now)

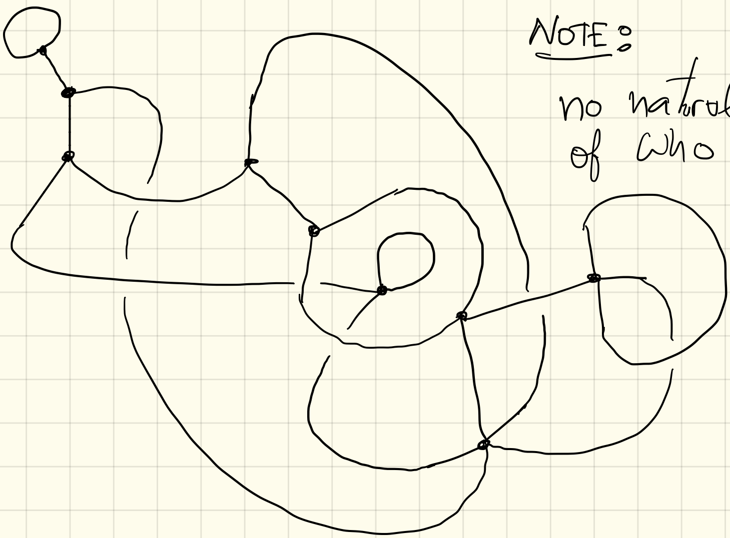
$$\sigma_x^{-1} = \sigma_x \qquad \frac{1}{2}(x\ y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\boxed{Z = \int dx \int dy\ e^{-xy + g\frac{x^3}{3!} + g\frac{y^3}{3!}}}$$



ALL VERTICES QUARTIC, AS IN THE FIG

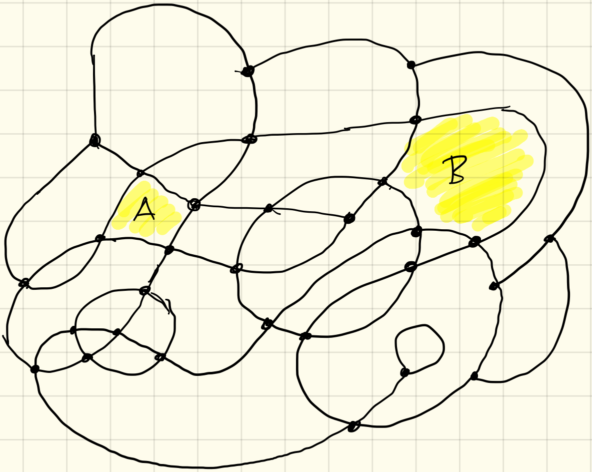
g] Exercise: Write down integral for this $\mathcal{Z}$



NOTE:

no natural notion of who is close. Can alway deform the graph preserving connections!

IF THE GRAPH HAD TO BE DRAWN ON A SURFACE THEN IT WOULD BE BETTER!



A & B far from each other in this complicated PLANAR graph

Looks Like a Discretization of a Random Surface $\longleftrightarrow$ 2D quantum gravity

Plane = Sphere (just add pt @ ∞)

Planar graps = Sphere graphs

can be drawn on plane without lifting pen

"..... sphere ....."

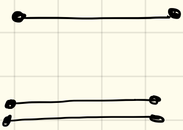
(For mathematitians graphs $\neq$ maps)

(* or maps)

Next: Graphs* and Topology

Fat graps

one index, vector



becomes

two indices, matrix

$$Z = \int \mathcal{D}M \exp(- \cdots)$$

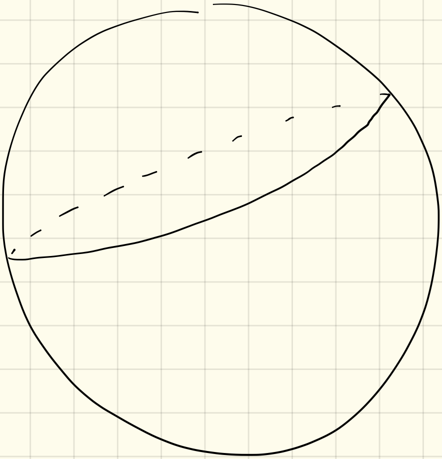
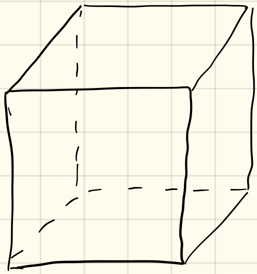
matrix

How to compute these

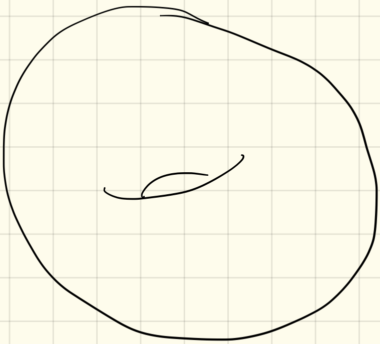
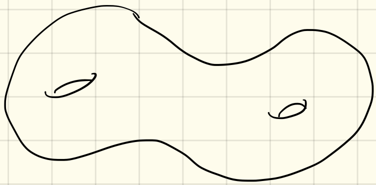
Some recent applications in high-energy physics

END OF LECTURE 2

Lecture 3

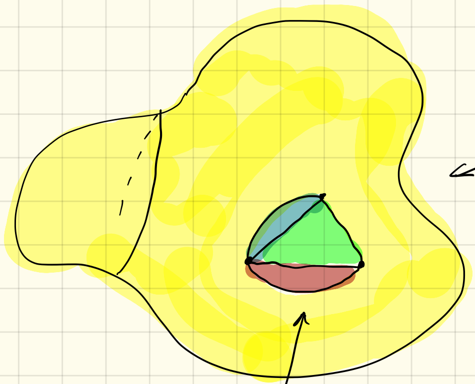
 $\cong$ 

as genus $g = 0$

 $g = 1$, 

has $g = 2$ etc

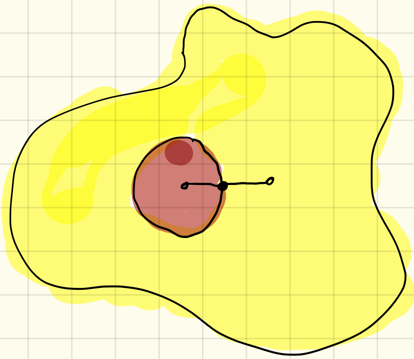
Deformation preserves genus $\longmapsto$ Topology



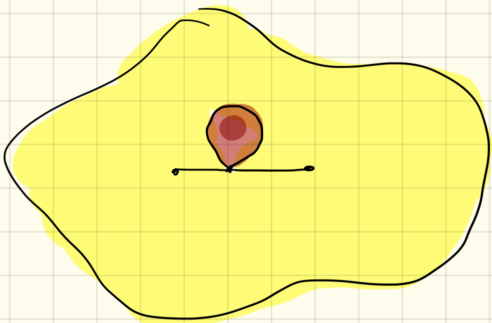
V = 3, F = 4,
E = 5

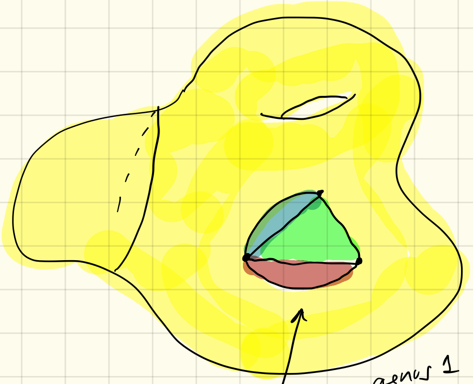
no edges intercept

map = graph drawn on a surface with all faces ≅ dics



$\neq$

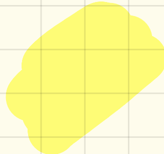




(it is a genus 1 map as seen before)

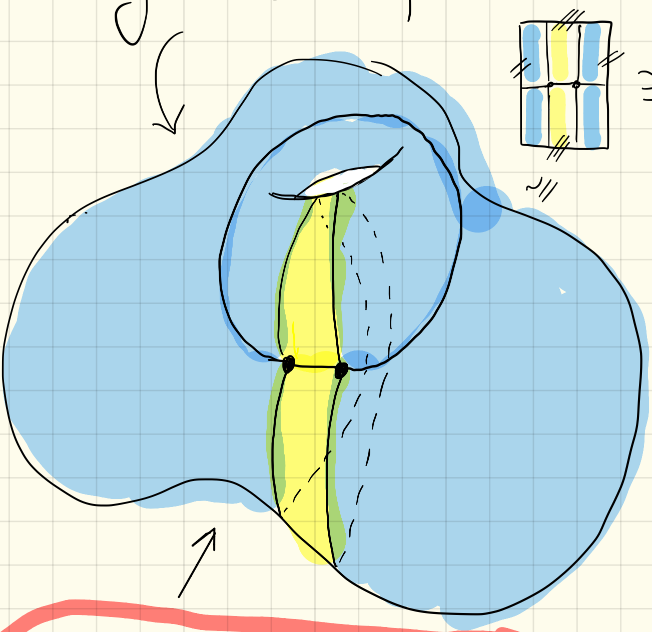
genus 1

not a map since

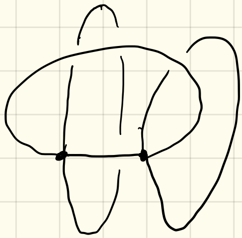


$\not\cong$ disc

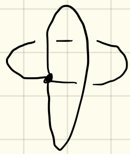
genus 1 map



$\cong$



or just



$V = 2, \ F = 2, \ E = 3$

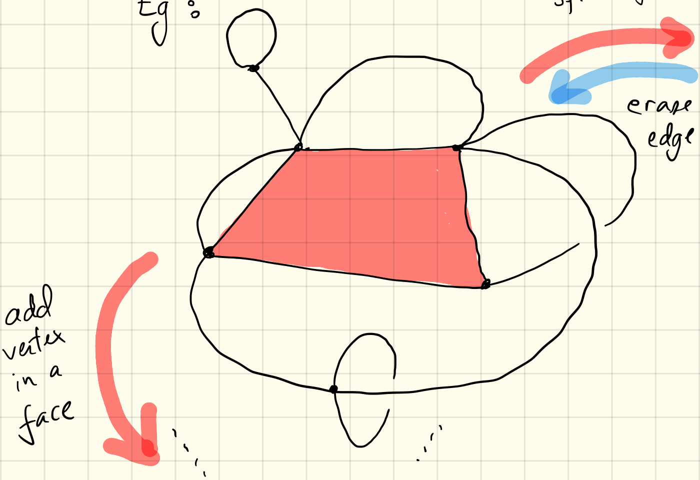
all genus 1 maps

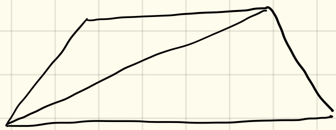
Consider a genus g map and consider

$$\chi \equiv V + F - E$$

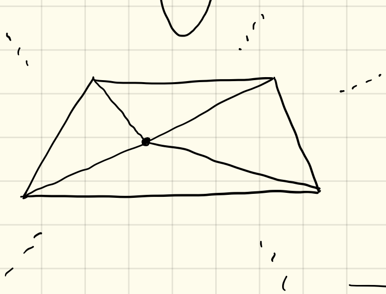
* it is a topological invariant!

Eg:





$$\begin{array}{r} \delta V = 0 \\ -(\delta E = 1) \\ \delta F = 1 \\ \hline 0 \end{array}$$



$$\begin{array}{r} \delta V = 1 \\ \delta F = n-1 \\ -(\delta E = n) \\ \hline 0 \end{array} \qquad (n=4)$$

more precisely

∃ beautiful proof with electric wires, one w/ spanning trees, angles on sphere, ...

$$\boxed{\begin{array}{l}\chi \equiv V + F - E = 2 - 2g \\ \forall \text{ connected genus } g \text{ map}\end{array}}$$

suffices to check on few examples and use topological invariance of previous page to prove more generaly by induction

Gluing also powerful:

$= \text{genus } g_1 + g_2$

$\chi_2 = 2 - 2g_2$

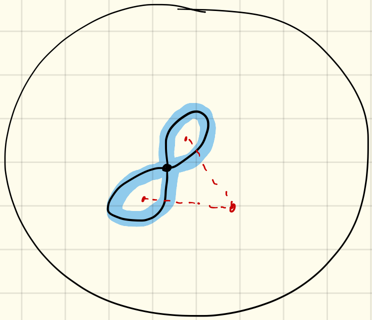
$\chi_1 = 2 - 2g_1$

$$\chi_{\text{glued}} = \chi_1 + \chi_2 \underbrace{- n}_{\substack{\text{shared}\\\text{vertices}}} \overbrace{- 2}^{\text{shared faces}} \underbrace{+ n}_{\substack{\text{shared}\\\text{edges}}}$$

$$= 2 - 2g_1 + 2 - 2g_2 - 2 = 2 - (g_1 + g_2) \blacksquare$$

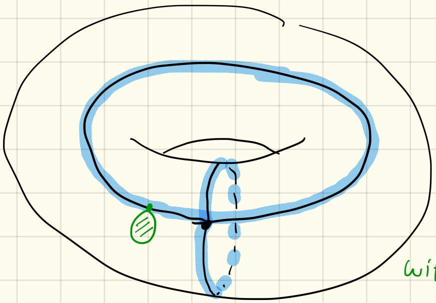
So checking $g=0$ and $g=1$ is enough

So, some examples



$F=3$, $V=1$, $E=2$

$F+V-E = 2 = 2-0$ ■



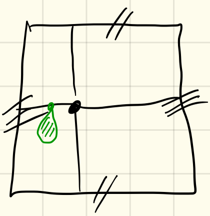
$F=1$, $V=1$, $E=2$

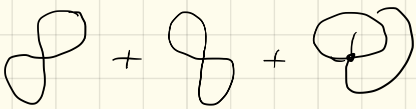
$F+V-E = 0 = 2-2\times 1$ ■

with $F=2$, $V=2$, $E=4$

$\chi = 0 = 2-2\times 1$ ■ ✓

2|



[before 

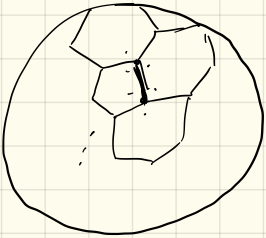
$= 3 = 3$!! but now

$3 = 2 + 1$ ← genus 1

↑ planar]

Aplication

Will not have time for this



Soccer ball with P pentagons and H hexagons

$$\left.\begin{aligned} F &= P + H \\ 2E &= 5P + 6H \\ 3V &= 5P + 6H \end{aligned}\right\} \quad V + F - E = \frac{5P+6H}{3} + P + H - \frac{5P+6H}{2}$$

see figure
(4 too much, 2 too few)

$$= P\underbrace{\left(\tfrac{5}{3} + 1 - \tfrac{5}{2}\right)}_{\frac{10+6-15}{6}} = \frac{P}{6}$$

H drops out!

$$= 2 \Rightarrow \boxed{P = 12}$$

[Works for fullerenes also, not just soccer balls]

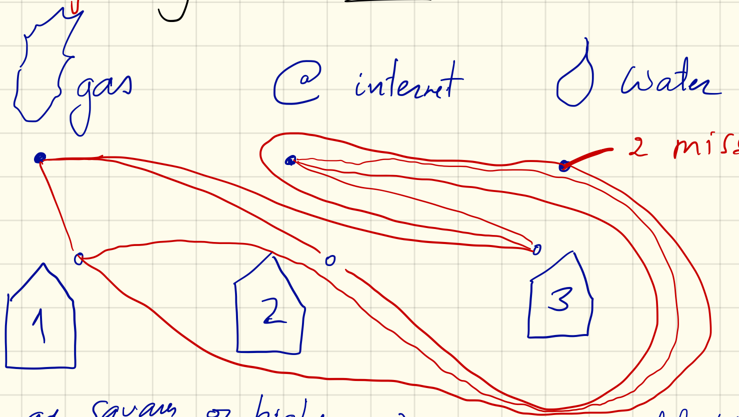
10] Exercise:

Can we use just pentagons?

Another Application

no time

Can 3 utilities be connected to 3 homes planarly? <u>No</u>:



2 missing!

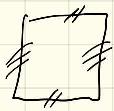
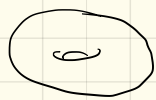
indeed not possible:

$$\left.\begin{aligned} E &= 9 \\ V &= 6 \\ g &= 0 \end{aligned}\right\} F=5$$

faces are squares or higher $\Rightarrow F=5 \Rightarrow 10$ edges at least!

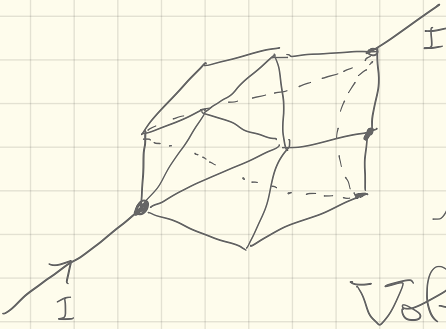
for $g = 1 \to F = 3 \to$ 2 squares, 1 decagon $\to 9 \times 2$!

11] Exercise: Draw torus config and identify the squares and decagon

Hint: Use  for torus, not !

Electric proof

no time



planar map

Voltage drop @ each face = 0

↓ last automatic

$F - 1$ eqs

Current conserved at each vertex

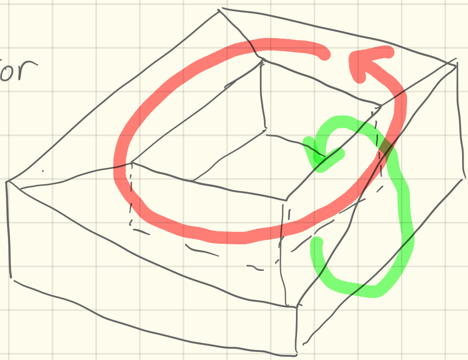
$V - 1$ eqs

last automatic

E currents to find $= V - 1 + F - 1$

$= V + F - 2$ ∎

for



2 more eqs for no voltage drop $\Rightarrow$

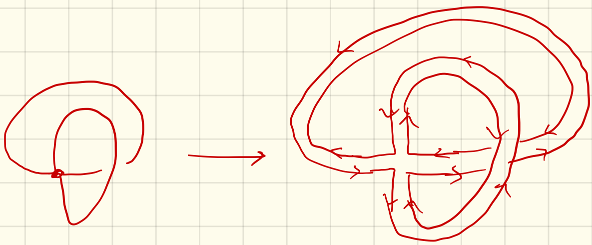
$E = V + F - 2 + 2$ ∎

etc

So, if we want to keep track of genus we need only keep track of F since E and V are trivial to follow!

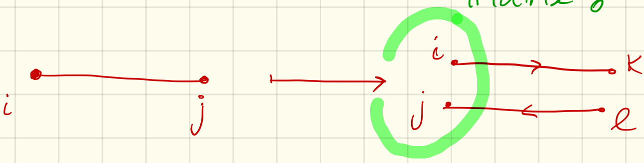
Idea

Sum over loops $\longmapsto N^3$ 3 faces



One colour loop only $\longmapsto N^1$ 1 face

$\widehat{\text{Matrix}}$!



$= \delta_{ij}\,\delta_{kl}$

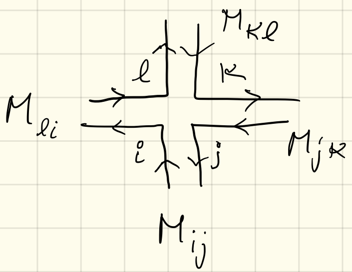
with $i,j,k,l = 1,2,\ldots,N$

LECTURE 4

$$Z \equiv \int \mathcal{D}M \, \exp\left(-\mathrm{tr}\, M^2/2 + g\, \mathrm{tr}\, M^4\right) \Big/ (\ldots)_{g=0}$$

$\mathcal{D}M$: $\underbrace{\text{hermitian}}$ $N\times N$ matrices

$\mathrm{tr}\, M^4$: $M_{ij} M_{jk} M_{kl} M_{li}$



$\underbrace{\qquad\qquad}$ good!

$$M_{jk}^* = M_{kj} = R_{kj} + i I_{kj}$$

$$M_{jk} = R_{kj} - i I_{kj}$$

symmetric

$$\mathrm{tr}\, M^2 = M_{ij} M_{jk}$$

$$= \sum_{i=1}^{N} M_{ii}^2 + 2 \sum_{j<k} \left(R_{jk}^2 + I_{jk}^2 \right)$$

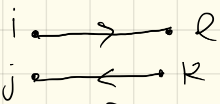
real

$$\Big\langle \underbrace{\left(R_{ij} + i I_{ij}\right)}_{M_{ij},\ i<j} \underbrace{\left(R_{kl} \pm i I_{kl}\right)}_{M_{kl}} \Big\rangle = 0 \text{ for } k<l$$

$+$: $k<l$; $-$: $k>l$

$$= \left(\tfrac{1}{2} + \tfrac{1}{2}\right) \times \delta_{il}\, \delta_{jk} \text{ otherwise}$$

as wanted!

- $\langle M_{ij} M_{kl} \rangle =$ 

$= \delta_{il}\, \delta_{jk}$!

good!

in formulas

$$\langle \operatorname{tr} M^4 \rangle = \langle M_{ij} M_{jk} M_{kl} M_{li} \rangle$$

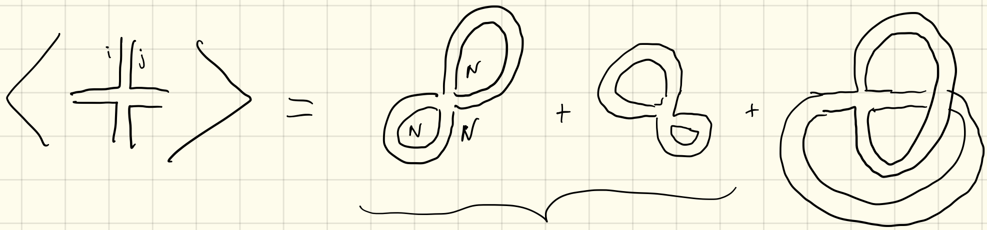
$$= MMMM + MMMM + MMMM$$

$$= (\delta_{ik}\underbrace{\delta_{jj}}_{N})(\delta_{ki}\underbrace{\delta_{ll}}_{N}) + \cdots + (\delta_{il}\delta_{jk})(\delta_{ji}\delta_{kl})$$

$$= N^2 \underbrace{\delta_{ii}}_{N} + N^3 + \underbrace{\delta_{ii}}_{N} = 2N^3 + N,$$

our much wanted $3!! = 2+1$!

in pictures



$$\underbrace{\underbrace{\quad}_{2N^3} \qquad \underbrace{\quad}_{N}}$$

genus 0   genus 1

$$Z = \dots + g^V N^F \# + \dots$$

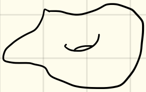
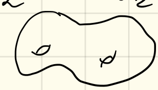
but $V + F - E = V + F - \frac{4V}{2} = 2 - 2g$

$$\Rightarrow F = 2 - 2g + V$$

$$Z = \dots + N^{2-2g} \Big(\underbrace{\overbrace{gN}^{\equiv \lambda}}_{\text{'t Hoof coupling '79}} \Big)^V + \dots$$

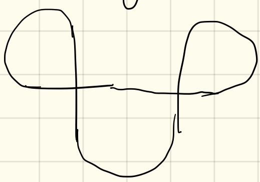
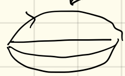
graph of genus g and V vertices

$$\log Z = \sum_{\text{genus } g = 0,1,\dots} N^{2-2g} \left[\sum_{\substack{\text{connected} \\ \text{graphs of} \\ \text{genus } g \\ \text{and } V \\ \text{vertices}}} \lambda^V \equiv \mathcal{F}_g(\lambda) \right]$$

$= N^2$ $\mathcal{F}_0$  $+ N^0$ $\mathcal{F}_1$  $+ N^{-2}$ $\mathcal{F}_2$  $+ \dots$

Sum surfaces like a string theory

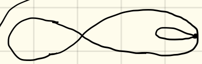
$V=2,\ g=0$

 $+$ 

who this one connects to

$$4\times 4\times 2 \quad + \quad 4 \quad = \boxed{36} \longrightarrow 18$$

$\frac{1}{2!}$ from exponent



first clockwise to pair in each vertex

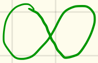
12] Exercise: Are there $g=2$ graphs with $V=2$?

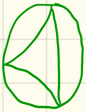
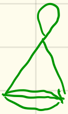
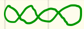
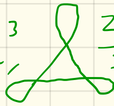
13] Exercise: How many $g=1$ are there?

Hint 1: See lecture 1!

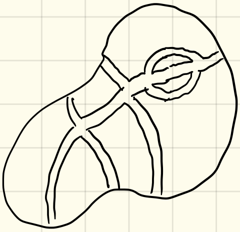
(the # is a # people often celebrate for birthday very seriously)

Amazingly one can often compute F_g!

 2λ,  $2\lambda^2$,  $16\lambda^2$

 $\frac{32}{3}\lambda^3$,  $64\lambda^3$,  $128\lambda^3$,  $\frac{256}{3}\lambda^3$

$288\lambda^3$ in total



$$= F_0(\lambda) = \frac{(u-1)(9-u)}{24} - \frac{\log u}{2}$$

$$\text{Where } u \equiv \frac{1-\sqrt{1-48\lambda}}{24\lambda} = 1 + 12\lambda + 288\lambda^2 + \dots$$

∞ previous page ?   See above!

$$-F_0(\lambda) = 2\lambda + 18\lambda^2 + 288\lambda^3 + \dots$$

$$= \sum \lambda^V \underbrace{\frac{12^V (2V-1)!}{V!\,(V+2)!}}_{\#\text{ graphs with } V \text{ vertices ?}}$$

$$\boxed{F_1 = \frac{\log(2-u)}{12}}$$

etc

How are such beautiful results derived?

1] $S[M] = S[\Lambda M \Lambda^{-1}]$ (Kind of gauge sym)

$$\Downarrow$$

eigenvalues of M

$$S[M] = S[z_j] = \frac{1}{2}\sum z_j^2 - \frac{\lambda}{N}\sum z_j^4$$

2] $$\mathcal{D}M = \prod_i dM_{ii} \prod_{i<j} d\,\mathrm{Re}\,M_{ij} \prod_{i<j} d\,\mathrm{Im}\,M_{ij}$$

$O(N^2)$ variables

$$= \underbrace{(\text{Jacobian})} \prod_{i=1}^{N} dz_i$$

$O(N)$ var!

$$= \Delta^2(z) = \prod_{i<j} (z_i - z_j)^2$$

Vandermond Determinant

3] Action is huge $\Rightarrow$ extremum dominates

$$Z \sim e^{-S_{cl}}$$

Where

$$S_{cl} = -\sum z_i^2/2 + \sum \frac{\lambda}{N} z_i^4 + \sum_{i \neq j} \log |z_i - z_j|$$

Where z_i obey extremum condition $\frac{\partial S}{\partial z_i} = 0$ or

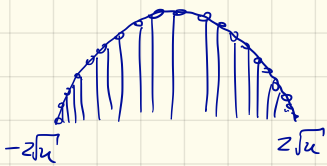
$$\frac{1}{2} z_i - 2\frac{\lambda}{N} z_i^3 = \sum \frac{1}{z_i - z_j}$$

2D Coulomb like problem!

repulsion

external force

4] replace $z_i \longrightarrow \rho(z)$ since N is huge



previous eq $\rightarrow$ int eq for ρ with solution

$$\rho(z) = \left[1 - 8\lambda u - 4\lambda^2\right]\sqrt{4u - \lambda^2}$$

defined above

5] plug this density into S_{cl} to get

$$S_{cl} \simeq -N^2 \mathcal{F}_0(\lambda) \text{ with } \mathcal{F}_0 \text{ as}$$

given above!

This is it about 1 MM (Matrix Model)

We can get more interesting decorated graps with 2MM, 3MM, ... like in lectures 1 and 2 with the colours.

$$\mathcal{Z} \equiv \int \mathcal{D}M\,\mathcal{D}N\, e^{-\operatorname{tr}\,[M\ N]\begin{bmatrix} /// & /// \\ // & // \end{bmatrix}\begin{bmatrix} M \\ N \end{bmatrix} + V(M) + V(N)}$$

still solvable (Kazakov e.g.)!

harder since $\operatorname{tr}(MN) \neq f(z_i, w_j)$

but still doable

$$\mathcal{Z}_{line} = \int \mathcal{D}M_1 \mathcal{D}M_2 \ldots \mathcal{D}M_n\, e^{-S}$$

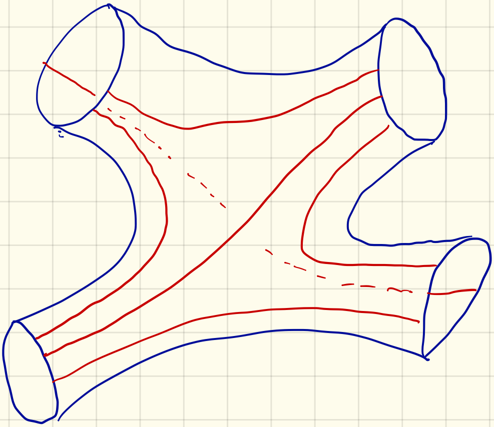
$$S = \operatorname{tr}(M_1 M_2) + \operatorname{tr}(M_2 M_3) + \ldots + \operatorname{tr}(M_{n-1} M_n)$$

$$+ \sum V(M_j) \quad \text{solvable !}$$

$$+ \operatorname{tr}(M_n M_1) \quad \underline{\text{not yet}} \text{ !!}$$

$\mathcal{Z}_{ring}$

String Theory



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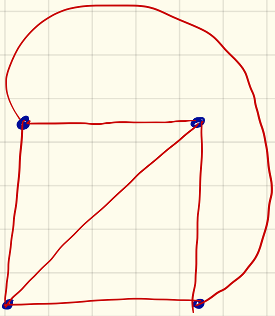
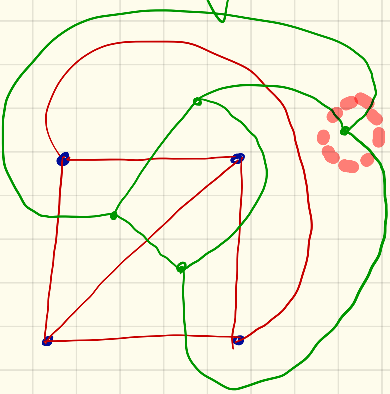


Triangulation

duality

Dual graph

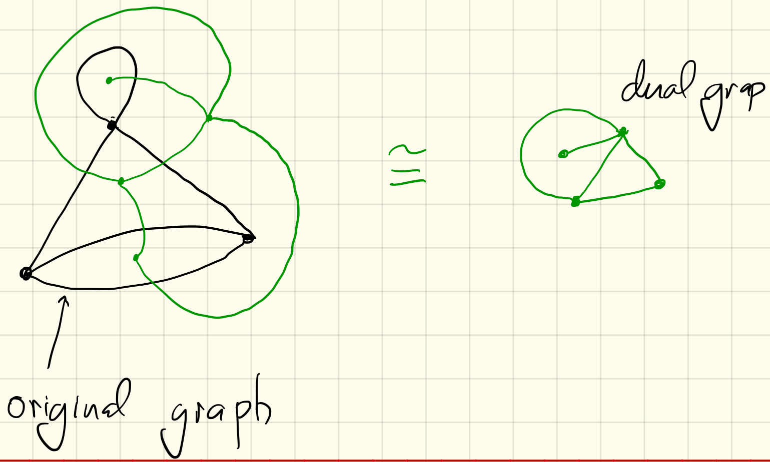


cubic vertices $\rightsquigarrow$ $g \,\mathrm{Tr}\, M^3$ model

triangle or hexagon coupling.

$\exists$ few such couplings $\mapsto$ $\mathbb{Z}_{\text{ring}}$
exactly the (still) unsolved one.

There is work to do !?



14] Exercise: What is the dual of a dual graph?

15] Exercise: What is the relation between (V, F, E, g) and $(\tilde{V}, \tilde{F}, \tilde{E}, \tilde{g})$ of the dual graph?

16] Exercise: What matrix model describes (the dual of) quadrangulations?

17] Exercise: We saw vectors x_i and matrices M_{ij}. What could tensors T_{ijk} etc be good for?

3 good references (plus review on the website):