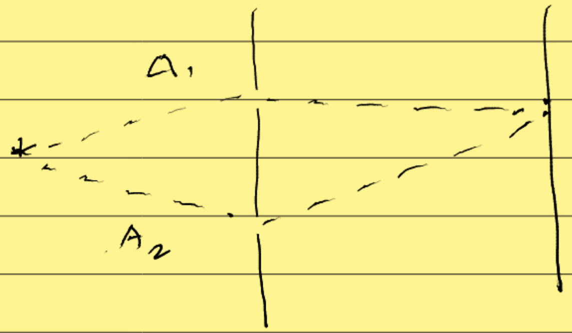
— Path Integral

$\downarrow$ integral over trajectories

Ideia : double-slit experiment (A. Zee)
(Q.M.)

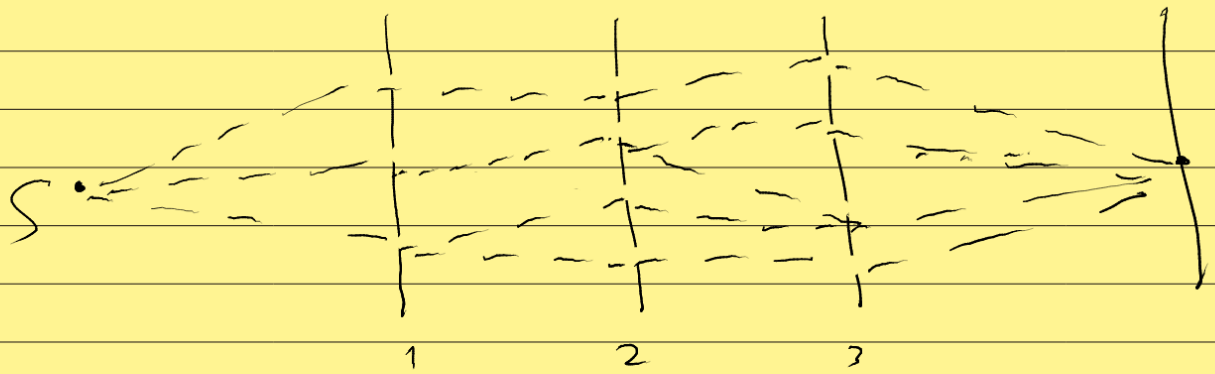


$O \leftarrow A_1 + A_2$ amplitude

detection probability at O

$\uparrow$ if more holes : $|A_1 + A_2 + A_3 + \dots|^2$

More screens with more and more holes :



Amplitude : $\sum_i A_i(S \to F_1 \to F_2 \to F_3 + \dots)$

In the limit of infinite number of screens, each with a "continuum of holes"

Amplitude from S to O =

$$= \sum \begin{pmatrix} \text{Amplitudes corresponding to} \\ \text{each trajectory} \end{pmatrix}$$

BUT, what is the probability amplitude of a given particle trajectory?

One can evaluate this as follows:

$$\Psi(q',t') = \int dq \; K(q',q;t'-t)\, \Psi(q,t) \qquad \text{--- (69)}$$

wave function

kernel (propagator)

$$K(q',q;t'-t) = \langle q' |\, e^{-i/\hbar\, \hat{H}(t-t')} \,| q \rangle$$

$$= \int \mathcal{D}q \; e^{i/\hbar\, S[q]} \qquad \text{--- (70)}$$

$S[q]$: classical action, $S = \int dt\, L(q)$

$\mathcal{D}q \to$ in the following

In the Schrödinger description of quantum mechanics, the state of a system is given by a vector $|\Psi(t)\rangle$ in a Hilbert space. The time evolution of $|\Psi(t)\rangle$ is governed by Schrödinger's equation:

$$i \frac{d|\Psi(t)\rangle}{dt} = \hat{H}|\Psi(t)\rangle \qquad \text{----(71)}$$

where $\hat{H}$ is the Hamiltonian of the system.

If the state of the system at an instant t is given by $|\Psi(t)\rangle$, the solution of Eq. (71) at time t' can be written formally as

$$|\Psi(t')\rangle = \hat{U}(t',t)\,|\Psi(t)\rangle \qquad \text{----(72)}$$

where $\hat{U}(t',t)$ is the evolution operator, given in terms of $\hat{H}$ as (for a time-independent $\hat{H}$)

$$\hat{U}(t',t) = e^{-i(t'-t)\hat{H}} \qquad \text{----(73)}$$

For simplicity, let's consider a system with just one degree of freedom q with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{q}) \qquad ----(74)$$

with

$$\hat{p} = -i\frac{d}{dq} \qquad \begin{cases} q: \text{ coordinate} \\ p: \text{ momentum} \end{cases} \qquad ----(75)$$

In the representation of the eigenstates of $\hat{q}$:

$$\hat{q}\,|q\rangle = q\,|q\rangle \qquad ----(76)$$

Eq. (2) can be written as

$$\langle q'|\psi(t')\rangle = \psi(q',t') = \langle q'|\hat{U}(t',t)\,|\psi(t)\rangle$$

$$= \int dq\ \langle q'|\hat{U}(t',t)|q\rangle\,\psi(q,t)$$

$$= \int dq\ K(q',q;t'-t)\,\psi(q,t) \quad --(77)$$

with

$$K(q',q;t'-t) = \langle q'|e^{-i(t'-t)\hat{H}}|q\rangle \quad --(78)$$

$\rightarrow$ propagator

being the <u>Kernel</u> of the evolution operator.

Note that Eq. (8) implies

$$\langle q'| e^{-i(t'-t)\hat{H}} |q\rangle = \langle q'| e^{-it'\hat{H}} e^{it\hat{H}} |q\rangle$$

$$= {}_H\langle q',t'| q,t\rangle_H \quad \text{----} (79)$$

where the index H means Heisenberg; in that the vector $|q,t\rangle_H$ is an eigenstate of the operator $\hat{q}$ in the Heisenberg description of quantum mechanics:

$$|q,t\rangle_H = e^{it\hat{H}} |q\rangle \quad \text{------} (80)$$

$$\hat{q}(t) = e^{it\hat{H}} \hat{q} \, e^{-it\hat{H}} \quad \text{-----} (81)$$

so that

$$\hat{q}(t)|q,t\rangle = e^{it\hat{H}} \hat{q} \, e^{-it\hat{H}} |q,t\rangle = e^{it\hat{H}} \underbrace{\hat{q}|q\rangle}_{q|q\rangle}$$

$$= q \, e^{it\hat{H}} |q\rangle = q\,|q,t) \quad \text{------} (82)$$

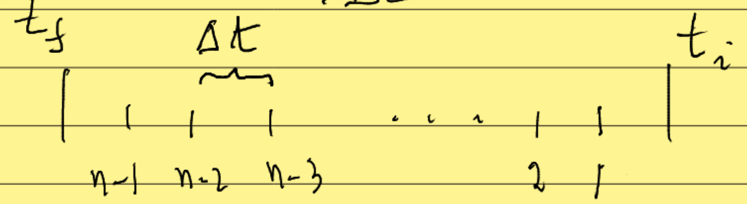
Therefore: $K(q',q;t'-t) = {}_H\langle q',t'|q,t\rangle_H$ is the probability amplitude of finding the system at the coordinate q' at time t' provided it was at the coordinate q at time t.

Feynman's path integral: express $K(q',q;t'-t)$ as a sum of trajectories $q_i(t)$ starting at time t at the coordinate q and ending at time t' at the coordinate q'

$\downarrow$

Let us divide the time interval in n pieces:

$$t_f - t_i = \underbrace{a}_{\Delta t}\, n \implies a = (t_f - t_i)/n \qquad \text{------ (83)}$$



$$\boxed{a = \Delta t}$$

and write

$$K(q',q;t'-t) = \langle q'|e^{-i(t'-t)\hat{H}}|q\rangle = \langle q'|e^{-i\overbrace{(a+a+\dots+a)}^{n}\hat{H}}|q\rangle$$

$$= \langle q'|e^{-ia\hat{H}}\, e^{-ia\hat{H}} \cdots e^{-ia\hat{H}}|q\rangle$$

$\int dq_{n-1}\,|q_{n-1}\rangle\langle q_{n-1}|$ $\qquad$ $\int dq_{n-2}\,|q_{n-2}\rangle\langle q_{n-2}|$ $\qquad$ $\int dq_1\,|q_1\rangle\langle q_1|$

$$= \int dq_{n-1}\cdots dq_1\, \underbrace{\langle q'|e^{-ia\hat{H}}|q_{n-1}\rangle \cdots \langle q_1|e^{-a\hat{H}}|q\rangle}_{n \text{ matrix elements}} \qquad \text{---- (84)}$$

Consider next one generic matrix element $\langle q_j|e^{-ia\hat{H}}|q_i\rangle$:

$$\langle q_j | e^{-ia\hat{H}} | q_i \rangle = \langle q_j | e^{-ia(\hat{H}_0 + \hat{V})} | q_i \rangle \quad \text{---(85)}$$

where

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} \quad \text{and} \quad \hat{V}(q) = V(q) \quad \text{---(86)}$$

For $n \to \infty$, a is infinitesimal, and

$$e^{-ia\hat{H}} = e^{-ia\hat{V}/2}\, e^{-ia\hat{H}_0}\, e^{-ia\hat{V}/2} + O(a^3) \quad \text{----(87)}$$

and one can write

$$\langle q_j | e^{-ia\hat{H}} | q_i \rangle = \langle q_j | e^{-iaV/2}\, e^{-ia\hat{H}_0}\, e^{-iaV/2} | q_i \rangle$$

$$= e^{-ia[V(q_j)+V(q_i)]/2} \langle q_j | e^{-ia\hat{H}_0} | q_i \rangle$$

and ----(88)

$$\langle q_j | e^{-ia\hat{H}_0} | q_i \rangle = \langle q_j | e^{-ia\hat{p}^2/2M} | q_i \rangle$$

$$= \int dp\, \langle q_j | \underbrace{e^{-ia\hat{p}^2/2M} | p \rangle}_{e^{-iap^2/2M} | p \rangle} \langle p | q_i \rangle$$

$$= \int dp\, e^{-iap^2/2m} \underbrace{\langle q_j | p \rangle \langle p | q_i \rangle}_{\frac{1}{2\pi} e^{ip.(q_j - q_i)}}$$

$$= \int \frac{dp}{2\pi} \, e^{-i a p^2/2M + i p \cdot (q_j - q_i)}$$

$$= \left(\frac{M}{2\pi i a} \right)^{1/2} e^{i \frac{M}{2a} (q_j - q_i)^2} \quad \text{-----} (89)$$

Use this result in Eq. (14):

$$K(q', q; t'-t) = \left(\frac{M}{2\pi i a} \right)^{1/2} \int dq_1 \cdots dq_{n-1} \exp \left\{ - \frac{iM}{2a} \left[(q' - q_{n-1})^2 + \cdots + (q_2 - q_1)^2 + (q_1 - q)^2 \right] - a \left[1/2 \, V(q') + V(q_{n-1}) + \cdots + V(q_1) + V(q) \right] \right\}$$

$$\text{------} (90)$$

Note that:

$$\tfrac{1}{2} V(q') + V(q_{n-1}) + \cdots + V(q_2) + V(q_1) + \tfrac{1}{2} V(q) =$$

$$= \tfrac{1}{2} \left[V(q') + V(q_{n-1}) \right] + \cdots + \tfrac{1}{2} \left[V(q_2) + V(q_1) \right]$$

$$+ \tfrac{1}{2} \left[V(q_1) + V(q) \right] \quad \text{----------} (91)$$

$$\downarrow$$

$$\frac{1}{2} \left[V(q_j) + V(q_i) \right] \simeq V(\bar{q}_i) + O(a^2) \text{ --}$$

$$\bar{q}_j = 1/2 (q_j + q_i), \quad q_j = q(t_j) = q(t_i + a) \quad \text{---} (92)$$

$$V(q_j) = V(\bar{q}_i) + (q_j - \bar{q}_i) V'(\bar{q}_i) + 1/2 (q_j - \bar{q}_i)^2 V''(\bar{q}_i) + \cdots \quad \text{------} (93)$$

$$V(q_i) = V(\bar{q}_i) + (q_i - \bar{q}_i) V'(\bar{q}_i) + 1/2 (q_i - \bar{q}_i)^2 V''(\bar{q}_i) + \cdots \quad \text{---} (94)$$

$$q(t_j) = q(t_i + \ ) \simeq q(t_i) + a\,\dot{q}(t_i)$$

$$= q(t_i) + O(a) \qquad \text{------- (95)}$$

$$q_j - \bar{q}_i = q_j - \tfrac{1}{2}(q_j + q_i) = \tfrac{1}{2}(q_j - q_i) \simeq O(a)$$

$$q_i - \bar{q}_i = q_i - \tfrac{1}{2}(q_j + q_i) = \tfrac{1}{2}(q_i - q_j) \simeq -O(a)$$

$$\Downarrow$$

$$\tfrac{1}{2}\left[V(q_j) + V(q_i)\right] = V(\bar{q}_i) + O(a^2) \qquad \text{----- (96)}$$

Note also that:

$$\frac{1}{2Ma}(q_j - q_i)^2 = a\,\frac{1}{2M}\left(\frac{q_j - q_i}{a}\right)^2 \qquad \text{------- (97)}$$

and

$$\frac{q_j - q_i}{a} \xrightarrow[a \to 0]{} \dot{q}_i = \dot{q}(t_i) \qquad \text{------- (98)}$$

With these results, one can write the two terms in the exponent in Eq. (22) as:

1) $$\left[\tfrac{1}{2}V(q') + V(q_{n-1}) + \cdots + V(q_2) + V(q_1) + \tfrac{1}{2}V(q)\right]$$

$$= a\left[V(\bar{q}_{n-1}) + \cdots + V(\bar{q}_2) + V(\bar{q}_1) + V(\bar{q}_0)\right], \quad q_0 \equiv q$$

$$\longrightarrow \int_t^{t'} dt''\, V(q(t'')) \qquad \text{----- (99)}$$

2) $a \, {}^1\!/_{2M} \left[(\dot{q}_{n-1})^2 + \cdots + (\dot{q}_2)^2 + (\dot{q}_1)^2 + (\dot{q}_0)^2 \right]$

$$\longrightarrow \int_t^{t'} dt'' \, \frac{1}{2M} \left(\dot{q}(t'') \right)^2 \qquad \text{------} \quad (100)$$

Putting all together:

$$K(q', q; t'-t) = \int \mathcal{D}q \; e^{i S[q]} \qquad \text{-----} \quad (101)$$

with

$$S = \int_t^{t'} dt'' \Big[\underbrace{\frac{1}{2M} \dot{q}^2 - V(q)}_{\text{this is the Lagrangian}} \Big] \qquad \text{---} \quad (102)$$

and

$$\mathcal{D}q \equiv \lim_{\substack{n \to \infty \\ a \to 0}} \left(\frac{M}{2\pi i a} \right)^{n/2} \underbrace{dq_{n-1} \cdots dq_1}_{\prod_{i=1}^{n-1} dq_i} \qquad \text{---} \quad (103)$$

Note that

Integral that lead to result in Eq. (21) is not well defined: integrand oscillates at large $|p|$ and does not falls off rapidly enough to zero as $|p| \longrightarrow \infty$

$$\Downarrow$$

Make a Wick rotation $t \to -i\tilde{\tau}$ ----- (104)

$$\int \mathcal{D}q \; e^{i/\hbar \, S[q]} \longrightarrow \int \mathcal{D}q \; e^{-\frac{1}{\hbar} S_E(q)} \quad \text{----(105)}$$

$$S_E[q] = \int d\tau \left[\frac{1}{2m}\left(\frac{dq}{d\tau}\right)^2 + V(q) \right] \quad \text{----(106)}$$

↑ Euclidean action

In field theory one is interested in the case where $\Psi(q)$ is the ground state

$$\Psi_0(q) = \langle q \,|\, 0 \rangle$$

$|0\rangle$: vacuum

Quantities of particular interest are correlation functions → time-ordered products of fields

In Q.M. the equivalent would be, e.g.

$$\langle 0 | T[\hat{q}(t_1)\,\hat{q}(t_2)] | 0 \rangle \quad \text{--------- (107)}$$

↓ 2-point function

One can show (see Appendix to these notes) that:

$$\langle 0 | T[\hat{q}(t_1)\,\hat{q}(t_2)] | 0 \rangle \sim \frac{\delta}{\delta J(t_1)} \frac{\delta}{\delta J(t_2)} Z[J] \Big|_{J=0}$$

---- (108)

where

$$Z[J] = \int \mathcal{D}q \; e^{\frac{i}{\hbar} S_J[q]} \quad \text{------} \quad (109)$$

with

$$S_J[q] = \int dt \left[L(q,\dot{q}) + J(t)\, q(t) \right] \quad \text{---} \quad (110)$$

Field theory? What is the path integral?

Recall that:

$$q_\alpha(t) \longrightarrow \varphi(\vec{r},t)$$

$$\sum_\alpha \rightarrow \int d^4x \quad \text{-----} \quad (111)$$

Note that: we started with the set of masses connected by springs → continuum limit, dynamics of the medium described by a field $\varphi(\vec{x},t)$

$$\boxed{\begin{array}{l}\text{Continuum limit} \\ \equiv \text{"coarse graining"}\end{array}}$$

Upon quantization, phonons are the quanta of φ

As put by Zee:

In the realm of particle physics, one does not think of particles (like photons, ...) as being described by fields constructed from some "underlying" point masses tied together by springs

Ginzburg-Landau view

⇓ The modern view: $\mathcal{L}[\varphi]$ is constructed on the basis of symmetry: decide on the fields one wants to use to implement the wanted symmetry and write down the most general form for $\mathcal{L}[\varphi]$

"old times" no more than 2 time derivatives

EFT view: no restriction on the number of time derivatives, as we shall see shortly

Upon quantization → particles (photons, ...)

are the quanta of the field

$$Z = \int \mathcal{D}\varphi \; e^{iS[\varphi]} \qquad \text{------- (112)}$$

$$S[\varphi] = \int d^d x \; \mathcal{L}[\varphi]$$

contains terms like $\varphi^2, \varphi^4, --$

$$\mathcal{L}[\varphi] = \tfrac{1}{2}(\partial\varphi)^2 - V(\varphi) \qquad \text{---- (113)}$$

$\int \mathcal{D}\varphi$ ← understood as the discrete product, with fields defined on a spacetime lattice

Noninteracting scalar field

Introduce an external field $J(x)$:

$$\mathcal{L} = \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}m^2\varphi^2 + J\varphi \qquad \text{--- (114)}$$

Let us calculate $Z[J]$: $\hbar = 1$

$$Z[J] = \int \mathcal{D}\varphi \; e^{i\int d^4x \left\{ \frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2 \right] + J\varphi \right\}} \qquad \text{-----} (115)$$

First, let us rewrite the derivative term; we integrate by parts:

$$\int d^4x \, (\partial\varphi)^2 = \int d^4x \, \partial_\mu\varphi \, \partial^\mu\varphi$$

$$= \int d^4x \left[\partial_\mu(\varphi\,\partial^\mu\varphi) - \varphi \underbrace{\partial_\mu\partial^\mu}_{\partial^2}\varphi \right]$$

$$= -\int d^4x \, \varphi\,\partial^2\varphi + \text{surface } \underbrace{\cancel{\text{term}}}_{0} \qquad \text{-----} (116)$$

So that

$$Z[J] = \int \mathcal{D}\varphi \; e^{i\int d^4x \left[-\frac{1}{2}\varphi(\partial^2 + m^2)\varphi + J\varphi \right]} \qquad \text{-----} (117)$$

<u>Discretization:</u> $\quad |a| \qquad \ast \ \ast \ \ast \ \ast \ \ast \ \ast \qquad x = i\,a$

i: integer, a: lattice spacing

$\varphi(x) \to \varphi_i = \varphi(i\,a)$

$$\partial\varphi \to \frac{1}{a}(\varphi_{i+1} - \varphi_i) = \frac{1}{a}\sum_j \underbrace{M_{ij}}_{\text{matrix}}\varphi_j \qquad \text{---} (118)$$

$m^2\varphi^2 \to m^2\varphi_i^2$ $\qquad$ includes m^2

$\int d^4x \to a^4 \sum_i$ $\qquad$ $\bar{M}, \bar{J}$: absorb factors of a

$$Z[J] \to \lim_{N\to\infty} \int d\varphi_1 \dots d\varphi_N \; e^{\frac{i}{2}\sum_{ij}\varphi_i M_{ij}\varphi_j + \sum_i J_i \varphi_i} \quad \text{------ (119)}$$

integrals of this type

1) $$\int_{-\infty}^{+\infty} dx \; e^{-\frac{1}{2}ax^2 + ibx} = \left(\frac{2\pi}{a}\right)^{1/2} e^{-b^2/2a}$$

$$\Big\downarrow \quad a \to ia$$

$$\int_{-\infty}^{+\infty} dx \; e^{-\frac{1}{2}iax^2 + ibx} = \left(\frac{2\pi}{ia}\right)^{1/2} e^{ib^2/2a} \quad \text{------ (120)}$$

2) $$\int_{-\infty}^{+\infty} dx_1 \dots dx_N \; e^{-\frac{1}{2}x\cdot A\cdot x + i\, b\cdot x} =$$

$$= \int_{-\infty}^{+\infty} dx_1 \dots dx_N \; \exp\Big[-\tfrac{1}{2}\big(A_{11}x_1^2 + A_{22}x_2^2 + \dots + A_{12}x_1x_2 + \dots\big) + i\,(b_1x_1 + \dots + b_Nx_N)\Big]$$

exercise (see eq. Zee)

$$= (2\pi)^{N/2} \frac{1}{\sqrt{\det A}} \, e^{\frac{1}{2} J\cdot A^{-1} J} \quad \text{------ (121)}$$

$$\Big\downarrow$$

$a \to ia$, like previously

$$\int_{-\infty}^{+\infty} dx_1 \ldots dx_N \; e^{-\frac{i}{2} x \cdot A \cdot x + i J \cdot x} = \frac{(-2\pi i)^N}{\sqrt{\det A}} e^{\frac{1}{2} J A^{-1} J} \quad \text{-- (122)}$$

Coming back to Eq. (117), one sees that

$$A \longrightarrow \bar{M} \longrightarrow \underbrace{-(\partial^2 + m^2)}_{\text{we need its inverse } \equiv D} \text{ in the continuum}$$

$$\Downarrow$$

$$-(\partial_x^2 + m^2)\, D(x-y) = \delta^{(4)}(x-y) \quad \text{---- (123)}$$

Solve this by Fourier transform:

$$D(x-y) = \int \frac{d^4x}{(2\pi)^4} e^{ik.(x-y)} \tilde{D}(k)$$

$$\text{---- (124)}$$

$$\delta^{(4)}(x-y) = \int \frac{d^4x}{(2\pi)^4} e^{ik.(x-y)}$$

$\Downarrow$ Exercise

$$\boxed{\tilde{D}(k) = \frac{1}{k^2 - m^2 + i\varepsilon}} \quad \text{---- (125)}$$

$i\varepsilon$ $\rightarrow$ to avoid a pole

Note: pole, mass-shell condition

$k^2 = m^2 \therefore k^{0\,2} = \vec{k}^2 + m^2 \Rightarrow E^2 = \vec{k}^2 + m^2$

Integration of $\tilde{D}(k)$ to obtain $D(x-y)$:

– integrate first over k^0 using contour method

$\Downarrow$ exercise (see Zee)

$$D(x) = -i \int \frac{d^3k}{(2\pi)^3\, 2\omega_k} \left[e^{-ik\cdot x}\, \theta(x^0) + e^{ik\cdot x}\, \theta(-x^0) \right] \quad \text{--- (126)}$$

$$\begin{cases} k\cdot x = \omega_k t - \vec{k}\cdot\vec{x} \\ \omega_k = (\vec{k}^2 + m^2)^{1/2} \end{cases}$$

where

$$\theta(x^0) = \begin{cases} 1, & x^0 > 0 \\ 0, & x^0 < 0 \end{cases} \quad \text{------ (127)}$$

Physics of $D(x)$:

It is not difficult to show that

$$i D(x) = \langle 0 | T[\hat{\varphi}(x)\hat{\varphi}(0)] | 0 \rangle \quad \text{----- (128)}$$

with

$$\hat{\varphi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(e^{-ik\cdot x}\, \hat{a}_k + e^{ik\cdot x}\, \hat{a}_k^{\dagger} \right) \quad \text{---- (129)}$$

and

$$T[\hat{\varphi}(x)\hat{\varphi}(y)] = \theta(x^0 - y^0)\, \hat{\varphi}(x)\hat{\varphi}(y) + \theta(y^0 - x^0)\, \hat{\varphi}(y)\hat{\varphi}(x) \quad \text{------ (130)}$$

$D(x) \to$ amplitude for a "φ-particle" propagate from 0 to x

- $\varphi(0)|0\rangle \to$ creates a wave packet at 0
- $\hat{\varphi}(x)\left(\hat{\varphi}(0)|0\rangle\right) \to \langle 0|\hat{\varphi}(x)\hat{\varphi}(0)|0\rangle$

$\to$ annihilates the packet at x

Therefore, one can write for $Z[J]$, Eq. (117):

$$Z[J] = \overbrace{\text{const.}}^{\text{indep. of } J}\; e^{-\frac{i}{2}\int d^4x\, d^4y\, J(x)\, D(x-y)\, J(y)}$$

$$= \text{Const.}\; e^{iW[J]}$$

where

$$W[J] = -\frac{1}{2}\int d^4x\, d^4y\; J(x)\, D(x-y)\, J(y) \qquad \text{------ (132)}$$

But, const. $= Z[0] = Z \Rightarrow$

$$Z[J] = Z[0]\, e^{iW[J]} \qquad \text{------ (134)}$$

Note that:

$$\left.\frac{\delta^2}{\delta J(x)\,\delta J(y)}\, e^{iW[J]}\right|_{J=0} = -\frac{\delta}{\delta J(x)}\Bigg\{\frac{i}{2}$$

$$\times \int d^4x'\, d^4y' \left[\frac{\delta J(x')}{\delta J(y)}\, D(x'-y')\, J(y') \right.$$

$$\left.\left. + J(x')\, D(x'-y')\, \frac{\delta J(y')}{\delta J(y)}\right]\, e^{iW[J]}\right|_{J=0}$$

$\delta^{(4)}(x'-y)$

$$= -\left(\frac{i}{2}\right) 2 \frac{\delta}{\delta J(x)} \int d^4y' \, D(x-y') J(y') \, e^{iW[J]} \Big|_{J=0}$$

$$= -\left(\frac{i}{2}\right) 2 \, D(x-y) = -i \, D(x-y) \quad \text{----} \ (135)$$

On the other hand:

$$\frac{\delta^2 e^{iW[J]}}{\delta J(x) \, \delta J(y)} \Bigg|_{J=0} = \frac{1}{Z} \frac{\delta^2 Z[J]}{\delta J(x) \, \delta J(y)} \Bigg|_{J=0}$$

$$= \frac{1}{Z} \frac{\delta^2}{\delta J(x) \, \delta J(y)} \int \mathcal{D}\varphi \, e^{iS[\varphi] + i\int d^4x \, J(x) \varphi(x)} \Bigg|_{J=0}$$

$$= \frac{(i)^2}{Z} \int \mathcal{D}\varphi \, \varphi(x) \varphi(y) \, e^{iS[\varphi]} \quad \text{----} \ (136)$$

$$\Downarrow$$

$$\boxed{ i D(x-y) = \frac{\int \mathcal{D}\varphi \, \varphi(x) \varphi(y) \, e^{iS[\varphi]}}{\int \mathcal{D}\varphi \, e^{iS[\varphi]}} } \quad \text{---} \ (137)$$

Recall that $D(x-y) = -i \langle 0 | T[\varphi(x) \varphi(y)] | 0 \rangle$

This generalizes to more than two fields.

$$\langle 0 | T[\varphi(x_1) \dots \varphi(x_N)] | 0 \rangle = \frac{1}{Z} \int \mathcal{D}\varphi \, \varphi(x_1) \dots \varphi(x_N) \, e^{iS[\varphi]} \quad \text{------} \ (138)$$