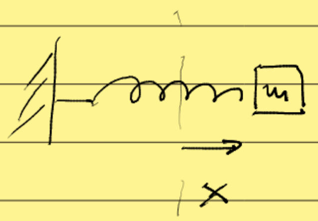
Flash course on QFT

$\left\{ \begin{array}{l} \text{canonical quantization} \\ \text{path integral} \end{array} \right.$

Canonical quantization:

Harmonic oscillator

Newton's law



$$m\ddot{x} = -kx \quad \therefore \quad \ddot{x} + \frac{k}{m}x = 0$$

$$\dot{x} = \frac{dx}{dt}, \quad \ddot{x} = \frac{d^2x}{dt^2} \qquad \boxed{\ddot{x} + \omega^2 x = 0}$$

$$x(t) = \cos\omega t,\ \operatorname{sen}\omega t,\ e^{ikx},\ e^{-ikx} \quad \text{----} \quad (17)$$

Or, equivalently:

Hamiltonian	Lagrangiana
$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$	$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \quad \text{-----} \quad (18)$
$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}$	$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$
$\ddot{x} = \frac{\dot{p}}{m} = -\underbrace{\frac{k}{m}}_{\omega^2}x$	$-kx - \frac{d}{dt}(m\dot{x}) = 0$

$$\ddot{x} + \omega^2 x = 0 \quad \text{---} \quad (19)$$

Quantization (Heisenberg representation)

$$x, p, H \longrightarrow \hat{x}, \hat{p}, \hat{H} = \frac{1}{2m}\dot{\hat{x}}^2 + \frac{1}{2}k\hat{x}^2$$

$$[\hat{x}, \hat{p}] = i\hbar \quad \text{-------- (20)}$$

$$\hat{O} = \hat{O}(\hat{x}, \hat{p})$$

$$\left.\begin{aligned} i\hbar \frac{d\hat{x}(t)}{dt} &= [\hat{x}(t), \hat{H}] \\ i\hbar \frac{d\hat{p}(t)}{dt} &= [\hat{p}(t), \hat{H}] \end{aligned}\right\} \quad i\hbar \frac{d\hat{O}(t)}{dt} = [\hat{O}(t), \hat{H}] \quad \text{-------- (21)}$$

Heisenberg equation

Future convenience: get rid of the mass m

1) Get rid of mass m

$$\left.\begin{aligned} \hat{x} &\equiv \hat{q}/\sqrt{m} \\ \hat{p} &= m\dot{\hat{x}} = \sqrt{m}\,\dot{\hat{q}} \end{aligned}\right\} \quad \hat{H} = \frac{1}{2}\dot{\hat{q}}^2 + \frac{1}{2}\omega^2\hat{q}^2 \quad \text{--- (22)}$$

2) Introduce ladder ops $\hat{a}, \hat{a}^\dagger$ $\to$ sol. Heisenberg eqs.

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2\omega}}\left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right) \quad \text{--- (23)}$$

$$\hat{p}(t) = \dot{\hat{q}}(t)$$

$$= -i\sqrt{\frac{\hbar\omega}{2}}\left(\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}\right) \quad \text{--- (24)}$$

$$\Downarrow$$

$$\boxed{\hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + 1/2\right)}$$

$$[\hat{q}, \hat{p}] = i\hbar$$

$$\Downarrow$$

$$[\hat{a}, \hat{a}^{\dagger}] = 1 \qquad \text{------ (25)}$$

3) $\hat{H}$ eigenstates:

Ground state: $\psi_0(q) = \underbrace{\langle q|\psi_0\rangle}_{\langle q|0\rangle} \sim e^{-\omega^2 q^2/2\hbar}$

$$\Downarrow$$

$$\hat{a}|0\rangle = 0 \qquad \text{---------- (26)}$$

$$\hat{H}|0\rangle = \hbar\omega\left(\hat{a}^{\dagger}a + 1/2\right)|0\rangle = \tfrac{1}{2}\hbar\omega|0\rangle$$

1st excited state: $\psi_1(q) = \underbrace{\langle q|\psi_1\rangle}_{\langle q|1\rangle} \sim x\, e^{-\omega^2 q^2/2\hbar}$

$$|1\rangle = \hat{a}^{\dagger}|0\rangle \qquad \text{-------- (27)}$$

$$\hat{H}|1\rangle = \hbar\omega\left(\hat{a}^{\dagger}a + 1/2\right)\hat{a}^{\dagger}|0\rangle$$

$$= 1/2\,\hbar\omega|1\rangle + \hbar\omega\,\hat{a}^{\dagger}\left(\overset{1}{[\hat{a}, \hat{a}^{\dagger}]} + \cancel{\hat{a}^{\dagger}\hat{a}}^{\,0}\right)|0\rangle$$

$$= 3/2\,\hbar\omega \qquad \text{---------- (28)}$$

$$|n\rangle \sim \hat{a}^{\dagger}|n-1\rangle \sim (\hat{a}^{\dagger})^n|0\rangle \sim \hat{a}|n+1\rangle$$

$$\hat{H}|n\rangle = (n+1/2)\hbar\omega|n\rangle$$

$$\hat{a}^{\dagger}\hat{a}|n\rangle = n|n\rangle, \quad n \text{ integer} \qquad \text{---- (29)}$$

Next: apply this to quantize the electromagnetic field in free space $\underset{\text{e.m.}}{\downarrow}$

Maxwell equations:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$$

ρ: charge density, $\vec{J}$: current

$$\rho = 0 = \vec{J}: \quad \underbrace{\nabla^2\vec{E} - 1/c^2\frac{\partial^2\vec{E}}{\partial t} = 0, \quad \nabla^2\vec{B} - 1/c^2\frac{\partial^2\vec{B}}{\partial t} = 0}_{\text{wave equation}}$$

For proper quantization

- introduce auxiliary fields $\vec{A}(\vec{r},t)$ and $\phi(\vec{r},t)$

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi, \qquad \vec{B} = \vec{\nabla}\times\vec{A} \qquad \text{------ (31)}$$

$\vec{A}$ is not unique: $\vec{A} \to \vec{A}$ -

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\alpha, \qquad \alpha = \alpha(\vec{r},t)$$

$$\phi \to \phi' = \phi - \frac{1}{c}\frac{\partial\alpha}{\partial t}$$

$\uparrow$ arbitrary function

$\Downarrow$

$$\underset{\sim}{\vec{E}'} = -\frac{1}{c}\frac{\partial \vec{A}'}{\partial t} - \vec{\nabla}\phi' = -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{A} + \vec{\nabla}\alpha\right)$$

$$-\vec{\nabla}\left(\phi - \frac{1}{c}\frac{\partial \alpha}{\partial t}\right) = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \cancel{\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\alpha}$$

$$-\vec{\nabla}\phi + \cancel{\frac{1}{c}\vec{\nabla}\frac{\partial \alpha}{\partial t}} = \underset{\sim}{\vec{E}}$$

Also, $\underline{\vec{B}' = \vec{B}} \rightarrow \vec{\nabla}\times(\vec{\nabla}\alpha) = 0$

Have to restrict $\vec{A}$ and ϕ, e.g. impose a "gauge" condition

$$\vec{\nabla}\cdot\vec{A} = 0 \qquad \underline{\text{Coulomb gauge}} \qquad \text{----} (32)$$

In this gauge, ϕ can be eliminated

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(-\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi\right) = -\nabla^2\phi = 4\pi\rho$$

↳ Poisson equation

$$\phi = \int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Also, in this gauge, for $\underline{\rho = 0 = \vec{J}}$:

$$\nabla^2\vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t} = 0$$

Put the system in a box:

$$\vec{A}(\vec{r},t) = \sum_{k} \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}}\, \vec{a}_{k}(t) \qquad \text{------} (33)$$

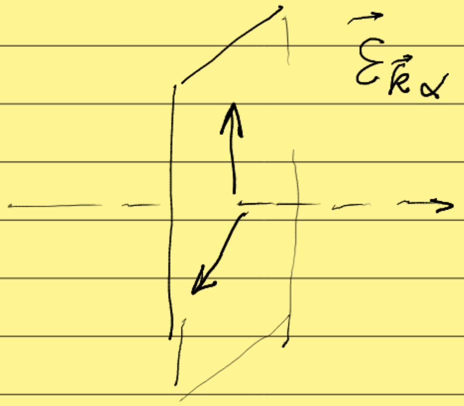
$\vec{\nabla} \cdot \vec{A} = 0 \Rightarrow \vec{k} \cdot \vec{a}_k(t) = 0$, independent of time

$$\Downarrow$$

$$\vec{a}_{\vec{k}}(t) \sim \vec{\varepsilon}_k \, q_k(t) \quad \therefore \quad \vec{k} \cdot \vec{\varepsilon}_k = 0 \qquad \text{----} (34)$$

↳ 2 components $\vec{\varepsilon}_{\vec{k}\alpha}$, $\alpha = 1, 2$

One possible choice:



$\vec{\varepsilon}_{\vec{k}\alpha}$ real,

$$\underbrace{\vec{\varepsilon}_{\vec{k}1} \times \vec{\varepsilon}_{\vec{k}2} = \hat{k} = \frac{\vec{k}}{|\vec{k}|} \qquad \vec{\varepsilon}_{\vec{k}\alpha} \cdot \vec{\varepsilon}_{\vec{k}\beta} = \delta_{\alpha\beta}} \qquad \text{---} (35)$$

$$\boxed{\vec{\varepsilon}_{\vec{k}\alpha} = \vec{\varepsilon}_{-\vec{k}\alpha}} \qquad \text{---} (36)$$

$$\vec{A}(\vec{r}, t) = \sum_{\vec{k}, \alpha} \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}} \, \vec{\varepsilon}_{\vec{k}\alpha} \, q_{\vec{k}\alpha}(t)$$

Replace this in wave equation Eq. (31):

$$\sum_{\vec{k}, \alpha} \left[\overbrace{|\vec{k}|^2 c^2}^{\omega_k^2} q_{\vec{k}\alpha}(t) + \ddot{q}_{\vec{k}\alpha}(t) \right] \cdot e^{i\vec{k}\cdot\vec{r}} = 0$$

$$\boxed{\ddot{q}_{\vec{k}\alpha}(t) + \omega_{\vec{k}}^2 \, q_{\vec{k}\alpha}(t) = 0} \qquad \text{-------} (37)$$

↑ harmonic oscillators

$\lambda \equiv (\vec{k}, \alpha)$

Quantization?

$$q_\lambda(t) \to \hat{q}_\lambda(t), \quad \hat{p}_\lambda(t) = \dot{\hat{q}}_\lambda(t)$$

Since $\vec{A}$ is real $\to q^*_\lambda(t) = q_{-\lambda}(t)$, $-\lambda \equiv (-\vec{k}, \alpha)$

$$\hat{q}^\dagger_\lambda(t) = \hat{q}_{-\lambda}(t) \qquad (38)$$

Expand in creation and annihilation operators

$$\hat{q}_\lambda(t) = \sqrt{\frac{\hbar}{2\omega_k}}\left(\hat{a}_\lambda e^{-i\omega_\lambda t} + \hat{a}^\dagger_\lambda e^{i\omega_\lambda t}\right)$$

$$(39)$$

$$\hat{p}_\lambda(t) = -i\sqrt{\frac{\hbar\omega_k}{2}}\left(\hat{a}_\lambda e^{-i\omega_\lambda t} - \hat{a}^\dagger_\lambda e^{i\omega_\lambda t}\right)$$

$$[\hat{a}_\lambda, \hat{a}^\dagger_{\lambda'}] = \delta_{\lambda\lambda'} \qquad (40)$$

Energy? Sum over energies of each oscillator

$$E = \sum_\lambda \hbar\omega_\lambda\left(\hat{a}^\dagger_\lambda \hat{a}_\lambda + 1/2\right) \qquad (41)$$

$$\hat{a}^\dagger_\lambda \hat{a}_\lambda |n\rangle = \overbrace{n}^{\text{integer}} |n\rangle$$

quanta

Energy sum over discrete "packages" of energy

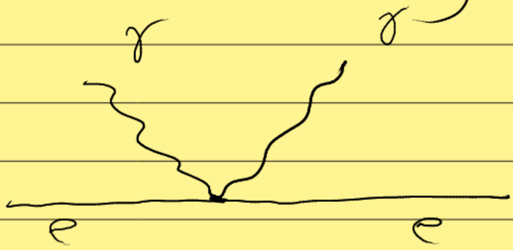
(also momentum, ...) photons (γ)

Interaction of photons with <u>nonrelativistic</u> particles: minimal coupling

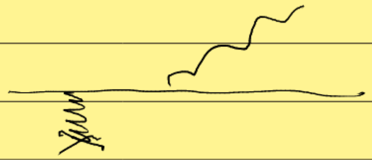
$$\hat{H} = \sum_i \frac{\hat{\vec{p}}_i^{\,2}}{2m} \rightarrow \sum_i \frac{1}{2m}\left(\hat{\vec{p}}_i - \frac{Q_i}{c}\vec{A}(\vec{r}_i)\right)^2$$

Typical processes:

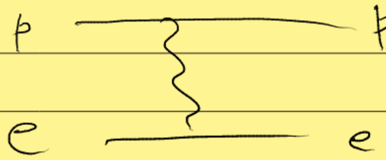
1) Thomson scattering: $e + \gamma \rightarrow e + \gamma$



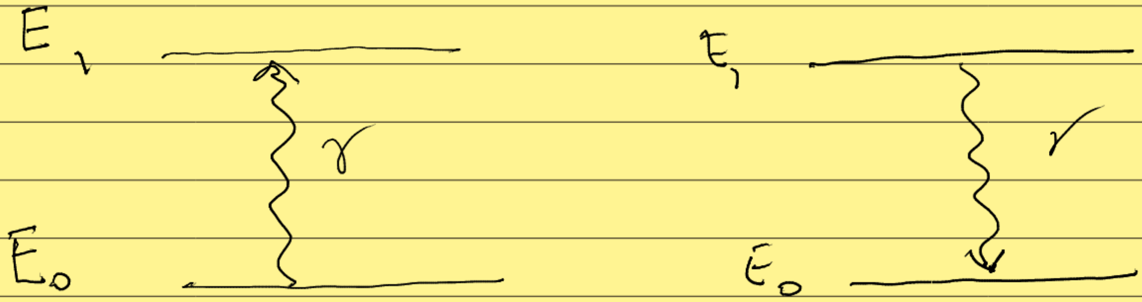
2) Bremstrahlung: $e \rightarrow e + \gamma$



3) Electron - proton interaction: $e + p \rightarrow e + p$



4) Radiative transitions in atoms:

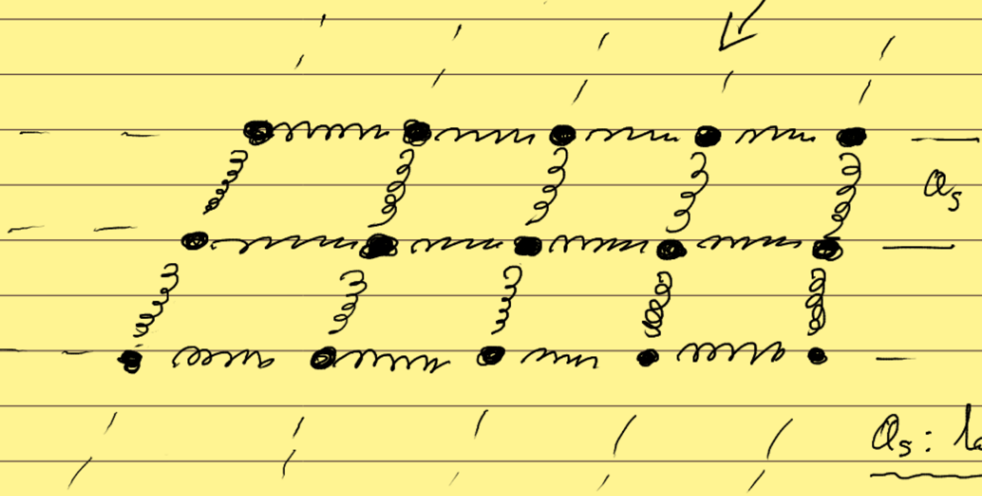


Scalar field : long wavelength (→ continuum) limit system of particles connected by springs [A. Zee]

⇓

2-dimensional lattice of point masses connected to each other by springs

Zee's mattress



a_s

a_s: lattice spacing

Consider the vertical (neglect the horizontal) displacement of the masses ⟵ $q_\alpha(t)$. The Lagrangian is

$$L = \frac{1}{2}\sum_{\alpha=1}^{N} M \dot{q}_\alpha^2(t) - V(q_1, \dots, q_N)$$

$$= \frac{1}{2}\sum_{\alpha=1}^{N} M \dot{q}_\alpha^2(t) - \underbrace{\frac{1}{2}\sum_{\alpha\beta}^{N} k_{\alpha\beta}\, q_\alpha q_\beta - \sum^{N} g_{\alpha\beta\gamma}\, q_\alpha q_\beta q_\gamma + \dots}_{} \qquad (4.2)$$

the q_α's are coupled to each other; as one mass moves, the neighboring ones move as well

only relative displacements give force ⇐

If one puts $g_{\alpha\beta\gamma} = 0$ (the harmonic approximation), the equation of motion is

$$M \ddot{q}_\alpha = - \sum_{\alpha\beta} k_{\alpha\beta} q_\beta \qquad \text{------ (43)}$$

Taking q_α as oscillating with frequency ω

$$q_\alpha(t) = e^{i\omega t} q_\alpha \qquad \text{--------- (44)}$$

one obtains

$$M\omega^2 q_\alpha = \sum_{\alpha\beta} k_{\alpha\beta} q_\beta \rightarrow \sum_\beta \left(M\omega^2 \delta_{\alpha\beta} - k_{\alpha\beta} \right) q_\beta = 0 \qquad \text{------ (45)}$$

Possible values of ω, the eigenfrequencies ω_i, are determined by the determinant $\| M\omega^2 - k \| = 0$.

The corresponding q_i ($\omega_i \rightarrow q_i$) are the eigenmodes:

Superposing eigemodes $\rightarrow$ <u>wave packets</u>

Upon quantization, they behave like particles – <u>the phonons</u>

This is similar to the quantization of electro-magnetic wave packets: <u>the photons</u>

Suppose we are interested in phenomena on length scales much greater than a_s (the lattice spacing) $\longrightarrow$ <u>take the continuum limit, i.e. $a_s \to 0$</u>

$$q_\alpha(t) \longrightarrow q(\vec{x},t) \Longrightarrow \varphi(x), \quad x = (\vec{x},t) \quad \text{-----} \; (46)$$

discrete   continuous

<u>Note that</u>: like "α" in q_α, "$\vec{x}$" in φ is a <u>label</u>, not a dynamical variable

Now, let $a_s \to 0$:

1) $$\tfrac{1}{2} M \sum_\alpha \dot{q}_\alpha = \tfrac{1}{2}\left(\frac{M}{a_s^2}\right) a_s^2 \sum_\alpha \frac{dq_\alpha}{dt}$$

$$\to \tfrac{1}{2}\,\sigma \int d^2x \; \frac{\partial \varphi(\vec{x},t)}{\partial t^2} \quad \text{---} \; (47)$$

2) $$V(q_1, \ldots, q_N) = \tfrac{1}{2} \sum_{\alpha\beta} k_{\alpha\beta}\, q_\alpha q_\beta + \ldots$$

$$k_{\alpha\beta}\, q_\alpha q_\beta = k_{\alpha\beta}\Big[\tfrac{1}{2}\,(q_\alpha - q_\beta)^2 - \tfrac{1}{2} q_\alpha^2 - \tfrac{1}{2} q_\beta^2\Big] \quad \text{-------} \; (48)$$

Now, for nearest-neighbor pairs: $\beta = \pm 1$ in $k_{\alpha\beta}$

$$(q_\alpha - q_\beta)^2 = a_s^2 \left(\frac{q_\alpha - q_\beta}{a_s}\right)^2$$

$$\rightarrow a_s^2 \left(\frac{\partial \varphi}{\partial x}\right)^2 \text{ or } a_s^2 \left(\frac{\partial \varphi}{\partial y}\right)^2 \qquad \text{---} (49)$$

depending on the direction joining the α and β sites

$\Downarrow$

Putting things together:

$$L\left(q_\alpha(t), \dot{q}_\alpha(t)\right) \longrightarrow \mathcal{L}\left[\varphi(x), \partial\varphi(x)\right]$$

$$\alpha, t \longrightarrow x, t \equiv x \qquad \text{-----} (50)$$

Action then becomes:

$$S[q] = \int dt\, L(q) \longrightarrow S[\varphi] = \int dt \int d^2x\, \mathcal{L}[\varphi] \qquad \text{------} (51)$$

e.g. φ^4

$$\mathcal{L}[\varphi] = \tfrac{1}{2}\sigma\left(\frac{\partial \varphi}{\partial t}\right)^2 - \tfrac{1}{2}\rho\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] - \tau\varphi^2 + \dots \qquad \text{----} (52)$$

where

$$\sigma \sim \frac{M}{a_s^2}, \quad \rho \sim k_{\alpha\beta}\, a_s^2, \quad \tau \sim k_{\alpha\beta} \qquad \text{----} (53)$$

Canonical Quantization (for now, $\tilde{c} = 0$)

$$\mathcal{L} = \frac{1}{2}\sigma\left(\frac{\partial\varphi}{\partial t}\right)^2 - \frac{1}{2}\rho\left(\vec{\nabla}\varphi\right)^2 \quad \text{------} \quad (54)$$

Eq. of motion: $q = q(t) \rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$

$\varphi = \varphi(\vec{r}, t)$ $\qquad \vec{r} = (x, y, z) \rightarrow x^i, \; i = 1, 2, 3$

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} - \frac{\partial}{\partial x^i}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial x^i}\right)} = 0$$

$$\boxed{\sigma\frac{\partial^2\varphi}{\partial t^2} - \rho\nabla^2\varphi = 0} \quad \text{------} \quad (55)$$

wave equation

$\sigma/\rho = v_s$ speed of sound

Quantization: expansion in Fourier series

$$\varphi(\vec{r}, t) = \sum_{\vec{k}} \varphi_k(\vec{r}, t) = \sum_{\vec{k}} q_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{r}} \quad \text{-----} \quad (56)$$

replace this in (55):

$$\frac{d^2 q_{\vec{k}}(t)}{dt} + \omega_k^2\, q_k(t) = 0, \quad \omega_k^2 = \rho/\sigma \cdot \vec{k}^2$$

harmonic oscillator

Situation similar to the e.m. case:

$$\varphi(\vec{r},t) = \sum_k f_k \left[\hat{a}_k e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + \hat{a}_k^\dagger e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)} \right] \qquad (57)$$

$f_k \leftarrow \sqrt{2\omega_k}$

Can obtain energy, momentum, ... of the elastic field: quanta are the <u>phonons</u>

Mattress → model for cristal lattice of ions of a material

+ electrons ⇒ <u>electron-phonon</u> interaction

in some situations, it is <u>attractive</u>, and <u>larger</u> than the repulsive electron-electron Coulomb interaction: net interaction is <u>attractive</u>

Formation of Cooper pairs ⇒ $\boxed{\text{Superconductivity}}$

Coulomb + phonon

e.m. field: relativistic, photons propagate with speed of light c:

Energy: $E_k = \hbar \omega_k = \hbar k c = p c$

Massive particles: $E_p^2 = p^2 c^2 + m^2 c^4$ ---- (58)

m: rest mass

as quanta of a field

Example: Scalar

Natural units: $\hbar = 1$, $c = 1$

$\hbar c \simeq 200$ MeV fm

$E_p^2 = p^2 + m^2 = k^2 + m^2$

$[E_p]$ = MeV or fm^{-1}

Lagrangian density:

$$\mathcal{L} = \tfrac{1}{2}\left(\frac{\partial \varphi}{\partial t}\right)^2 - \tfrac{1}{2}\left(\vec{\nabla}\varphi\right)^2 - \tfrac{1}{2} m^2 \varphi^2 \quad \text{---- (59)}$$

Eq. of motion (Euler-Lagrange):

$$\frac{\partial^2 \varphi}{\partial t^2} - \vec{\nabla}^2 \varphi + m^2 \varphi = 0 \quad \text{---- (60)}$$

Fourier decomposition:

$$\varphi \sim e^{i(\vec{k}\cdot\vec{r} - \omega_k t)}$$

replace this in Eq. (60)

$$-\omega_k^2 + \vec{k}^2 + m_0^2 = 0$$

$$\boxed{\omega_k^2 = \vec{k}^2 + m_0^2}$$ ← confirms Eq. (58)

Things are similar to the e.m. case

Covariant notation :

Space-time coordinates: $t, \ \vec{x} = (x, y, z)$ ↑ x^i

$t \equiv x^0 \Rightarrow x^\mu = (x^0, x^i), \quad \mu = 0, 1, 2, 3$

Space-time distance: $ds^2 = dt^2 - d\vec{x} \cdot d\vec{x}$

$= dt^2 - dx^i dx^i$ sum over repeated indices

metric tensor :

$g^{\mu\nu}$: $g^{00} = g_{00} = 1, \quad g^{ij} = -\delta^{ij} = g_{ij}$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} : \quad x^\mu = g^{\mu\nu} x_\nu \qquad \text{----} \quad (61)$$

$$ds^2 = g^{\mu\nu}\, dx_\mu\, dx_\nu = g_{\mu\nu}\, dx^\mu\, dx^\nu \quad \text{--- (62)}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x^0} = \partial_0 = \frac{\partial}{\partial x_0} = \partial^0 \quad \text{--- (63)}$$

$$\frac{\partial}{\partial x^i} = \partial_i = \underbrace{\nabla^i}_{\text{gradient}} = -\frac{\partial}{\partial x_i} = -\partial^i \quad \text{--- (64)}$$

$$\Downarrow$$

$$\frac{\partial^2}{\partial t^2} - \nabla^2 = g^{\mu\nu}\partial_\mu \partial_\nu = \partial_\mu \partial^\mu \equiv \partial^2 \quad \text{--- (65)}$$

Wave equation : $\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = 0 \;\therefore\; \boxed{\partial^2 \varphi = 0}$ --- (66)

Lagrangian density:

$$\mathcal{L} = \tfrac{1}{2}\left(\frac{\partial \varphi}{\partial t}\right)^2 - \tfrac{1}{2}\left(\vec{\nabla}\varphi\right)^2 - \tfrac{1}{2} m^2 \varphi^2$$

$$= \tfrac{1}{2}\partial_\mu \varphi\, \partial^\mu \varphi - \tfrac{1}{2} m^2 \varphi^2 = \tfrac{1}{2}(\partial \varphi)^2 - \tfrac{1}{2} m^2 \varphi^2 \quad \text{--- (67)}$$

Lagrangian :

$$L = \int d^3x\, \mathcal{L}$$

$$d^4x = dt\, d^3x$$

Action :

$$S = \int dt\, L = \int d^4x\, \mathcal{L} \quad \text{--- (68)}$$