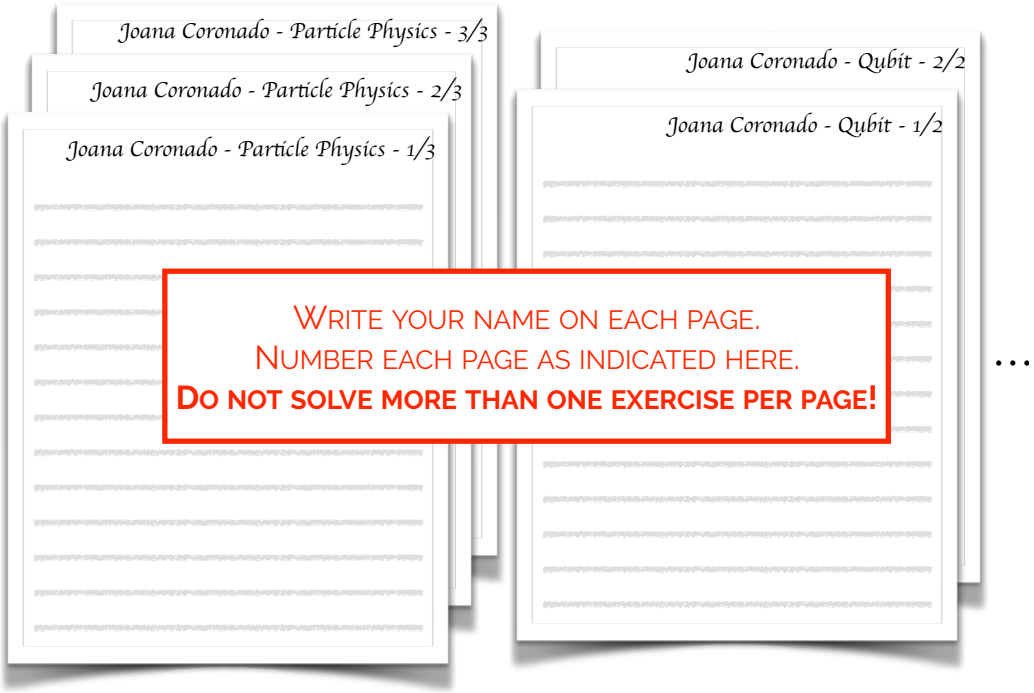
IFT-Perimeter-SAIFR
Journeys into Theoretical Physics
2016
Afternoon Exam

Important:



Scores:

- Problem 1 (Nodes in Quantum Mechanics): 33%
- Problem 2 (A Blue Sky): 19%
- Problem 3 (A Sticky Surface): 20%
- Problem 4 (Photographing a Relativistic Sphere): 28%

Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 Nodes in Quantum Mechanics

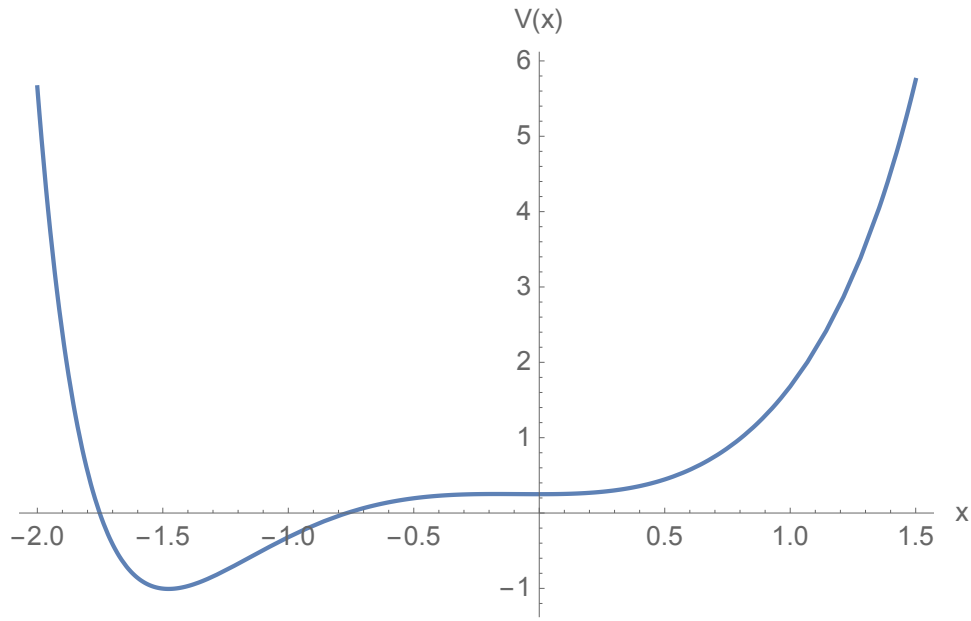


Figure 1: A smooth confining potential.

In this exercise we will explore the one dimensional Schrodinger equation

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\,\psi(x) \tag{1}$$

where $V(x)$ is some smooth potential tending to plus infinity as $x \to \pm\infty$ as depicted in figure 1.

Recall that we can think of (1) as a Sturm-Liouville problem. Namely we can always assume $\psi(x)$ vanishes as $x \to -\infty$. Then, for a generic E in (1) the wave function diverges at $x \to +\infty$. There is however a discrete set energies $E_0, E_1, E_2, \ldots$ for which the corresponding wave functions – denoted as $\psi_0, \psi_1, \psi_2 \ldots$ – also vanish at $x \to +\infty$ and are thus called normalizable wave functions. The E_n are the physical energies of the system and we will henceforth refer to the corresponding ψ_n as the physical solutions. In what follows we will (correctly) assume that they are smooth.

Warm-up

1. **[2pt]** Show that there is no degeneracy. That is, we can not have two different physical solutions ψ_1, ψ_2 with the same energy $E_1 = E_2$.

2. **[2pt]** Show that any physical solution can be taken to be a real function.

3. **[2pt]** Show that two physical solutions with different energies are orthogonal with respect to the norm $\langle\psi|\phi\rangle \equiv \int dx\, \psi(x)^*\phi(x)$.

   In what follows physical solutions are always normalized so that $\langle\psi_n|\psi_m\rangle = \delta_{nm}$.

4. **[2pt]** Consider now the energy functional

$$E[\psi] = T[\psi]+V[\psi]\,, \qquad T[\psi] \equiv +\frac{\hbar^2}{2m}\int dx\, |\psi'(x)|^2\,, \qquad V[\psi] \equiv \int dx\, V(x)|\psi(x)|^2\,. \tag{2}$$

   Show that for physical solutions $E[\psi_n] = E_n$.

5. **[2pt]** Consider a trial wave function $\psi(x)$. Since energy eigenstates for a complete base of states, we can always expand it out as $\psi(x) = \sum_n c_n\psi_n(x)$. Write down the condition on the c_n's so that ψ is normalized as $\langle\psi|\psi\rangle = 1$.

6. **[2pt]** Let E_0 be the ground state energy, i.e. $E_0 < E_1 < E_2 < \ldots$. Show that $E[\psi] \geq E_0$ for any normalized trial wave function.

Nodes

In this second part of the exercise we will establish that the n-th excited state ψ_n has exactly $n-1$ zeros in the real axis (plus a few other simple related facts).

It will often be convenient to use the results of the previous part. Feel free to use them even if you did not do that part of the exercise.

7. **[4pt]** At the end we will prove that the ground state wave function ψ_0 has no nodes in the real axis. For now establish the following simpler fact: We can have at most a single physical solution without nodes.

8. **[4pt]** Consider ψ to be a solution to (1) with energy E.

   As mentioned below (1), we define this wave function to vanish as $x \to -\infty$. Then, for a generic E it will diverge as $x \to +\infty$ unless E is a physical energy.

   Show that

$$\psi(x)\psi_n'(x) - \psi'(x)\psi_n(x) = \frac{2m}{\hbar^2}(E-E_n)\int_{-\infty}^{x} dy\, \psi(y)\,\psi_n(y) \tag{3}$$

9. **[4pt]** Consider now an energy E slightly larger than some E_n. Then ψ will be very close to ψ_n except that at very large x where it blows us while ψ_n vanishes.

   What is the sign of the right hand side of (3)?

10. **[4pt]** Now focus on the vicinity of a zero of ψ_n which we denote as $\psi_n(x_*) = 0$. Without loss of generality assume ψ_n is negative/positive to the left/right of x_*

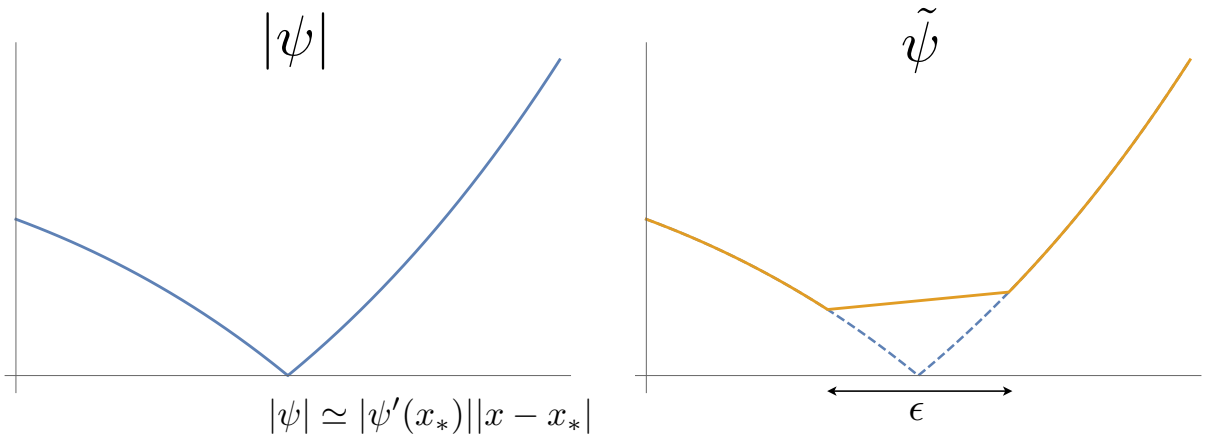


Figure 2: From a trial wave function with zeros we construct a new wave function without zeros by first constructing $|\psi|$ (on the left) and then a new function $\tilde{\psi}$ from that one by excising a small segment of length ϵ around each zero of ψ and replacing the wave function there by a linear interpolation (on the right). The resulting wave function $\tilde{\psi}$ is positive everywhere.

respectively such that $\psi'(x_*) > 0$. By continuity ψ should also change from negative to positive values in the neighbourhood of x_*.

Show that the zero of ψ is to the left of x_*.

You can now jump to the next point 11 and read the following comments later. (Nothing to solve here.) A picture is now clearly emerging about what goes on as we increase the energy smoothly all the way from one eigenstate with $E = E_n$ to the next with $E = E_{n+1}$. As we increase E all zeros move to the left. In particular, as soon as E bigger than E_n a new zero comes from $x = +\infty$ (where ψ_n vanishes) so that ψ has one more finite zero (at very large values of x) compared to ψ_n. By the point the energy reaches E_{n+1} that zero moved to some finite value. The wave function ψ (which is now equal to ψ_{n+1}) vanishes at infinity once again. As we increase E further we get one more zero coming form infinity and so on. To confirm this picture we can establish the following theorem: between any two consecutive zeros of a physical solution ψ_{n_1} there is a zero of ψ_{n_2} for any $n_2 > n_1$. To prove this, we denote the two consecutive zeros as x_L and x_R and show – basically following the same sort of reasoning leading to (3) – that

$$\psi'_{n_1}(x_R)\psi_{n_2}(x_R) - \psi_{n_1}(x_L)\psi'_{n_2}(x_L) = \frac{2m}{\hbar^2}(E_{n_2} - E_{n_1}) \int_{x_L}^{x_R} dy\, \psi_{n_1}(y)\, \psi_{n_2}(y) \qquad (4)$$

Then we assume that ψ_{n_2} does not change sign between x_L and x_R and reach a contradiction.

11. **[5pt]** Finally, the last missing element in the argument above is to show that the ground state wave function ψ_0 has no zeros in the real axis. This is what we turn to now. To establish this we turn to the energy functional (2).

    (a) Show that $E[\psi] = E[|\psi|]$.

    (b) Suppose the function ψ has some zeros. Then, the function $|\psi|$ is smooth except at its zeros where it has cusps as depicted in figure 2. Consider now a new function $\tilde{\psi}$ which is obtained from this function as indicated in figure 2. Note that $\tilde{\psi}$ has no zeros. We want to compute the variation $\delta E = E[\tilde{\psi}] - E[|\psi|]$. First we note that the kinetic energy dominates over the potential energy when computing this. Indeed,

        i. Explain why $\delta V \equiv V[\tilde{\psi}] - V[|\psi|] = \mathcal{O}(\epsilon^3)$

        ii. Explain why $\delta T \equiv T[\tilde{\psi}] - T[|\psi|] = \mathcal{O}(\epsilon)$

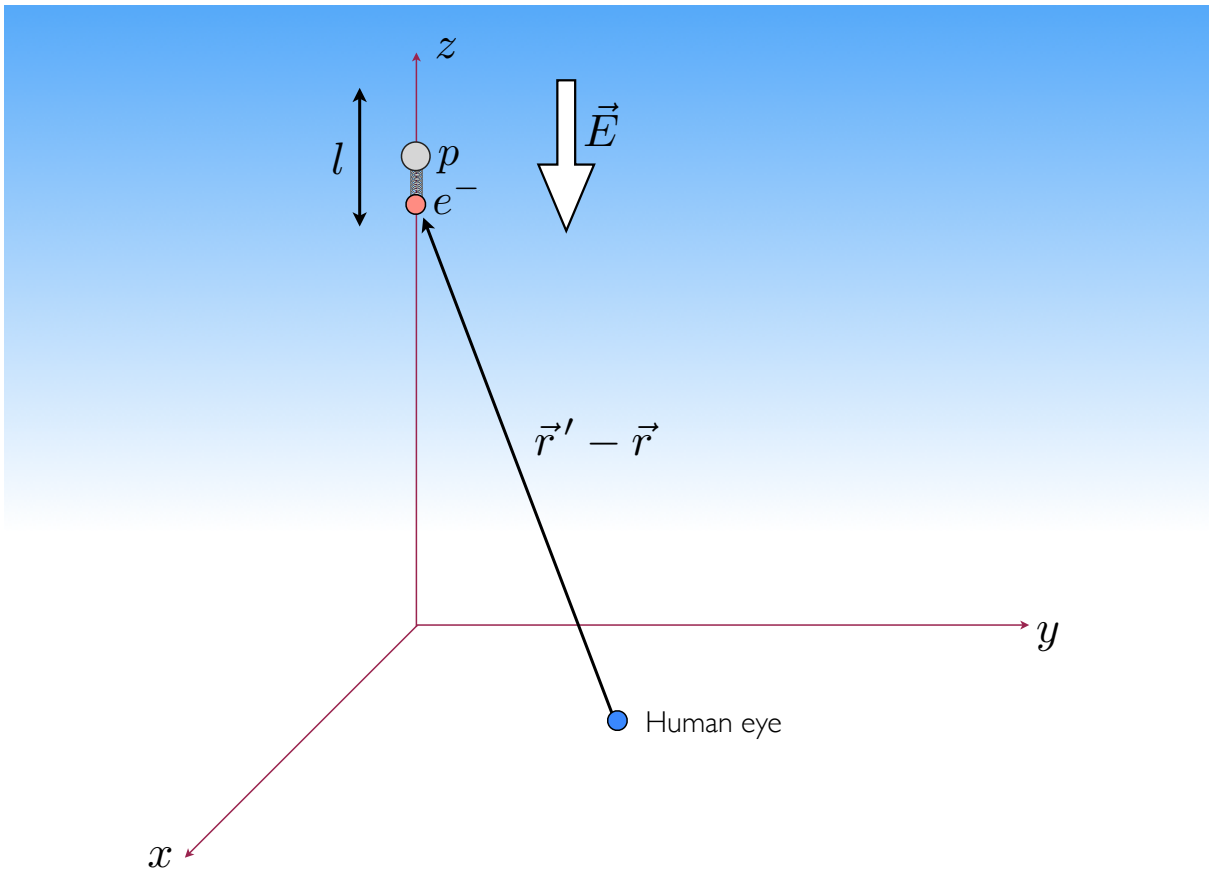
        iii. Show that δT is negative.

    (c) Explain why the ground state wave function ψ_0 has no zeros in the real axis.

And we are done. To summarize:

We learned that any smooth potential such as that in figure 1 we have a non-degenerate spectrum $E_0 < E_1 < E_2 \ldots$. The ground state wave function vanishes at infinity and has no nodes in the real axis. The first excited state ψ_1 has one finite zero. The second excited state ψ_2 has two zeros, one to the left of the zero of ψ_1 and one to its right. The third excited state ψ_3 has three nodes, one in between the zeros of ψ_2, one to their left and another one to their right. The next state has four zeros and so on. Basically, and in more physical terms, what we saw was that adding nodes entails a high cost kinetic energy-wise and this translates into these results.

2 A Blue Sky



In this question we will be concerned with the interaction of an electromagnetic wave with an atom, which we take to be a hydrogen atom, in the nonrelativistic limit. As it turns out, since the electron mass m_e is 2000 times smaller than the proton mass the dominant interaction comes from the electron and we will therefore focus on its response to the electromgnetic wave. We will model the hydrogen atom very crudely by a harmonic oscillator with a natural frequency denoted by ω_0.

Consider a plane electromagnetic wave with frequency ω polarized in such a way that the electric field is in the $\hat{z}$ direction and can be written as:

$$\vec{E} = E_0 \sin(\omega t)\ \hat{z}$$

1. **[2pt]** Write the equation of motion for the electron and show that the solution is ($-e$ is the electron charge):

$$z(t) = \frac{eE_0}{m(\omega^2 - \omega_0^2)} \sin(\omega t) \tag{5}$$

2. **[1.5pt]** Therefore the electron oscillates back and forth in the direction of the electric field and will emit radiation. We can think of the motion of the electron as a current produced by a charge $q(t) = -ez(t)$. Show that the associated current can be written as:

$$I(t) = I_0 \cos(\omega t) \tag{6}$$

What is I_0?

3. **[3pt]** We now want to compute the vector potential generated by the oscillation of the electron far away from the atom using the retarded potential:

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}, t - |\vec{r'} - \vec{r}|/c)}{|\vec{r'} - \vec{r}|} d^3r' \tag{7}$$

Writing $\vec{J}(\vec{r}, t) = I(t)\hat{z}\delta(x)\delta(y)$, noting that $\vec{r'} = z'\hat{z}$ in this case and assuming that the electron moves in the range $-l/2 < z < l/2$ show that in the limit $r \gg l$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} l \frac{I(t - r/c)}{r} \hat{z} \tag{8}$$

which has a typical $1/r$ behaviour of a radiation field.

4. **[3pt]** Using the Lorentz gauge condition which in the units we are using reads:

$$\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \Phi}{\partial t} = 0 \tag{9}$$

show that the scalar potential far away from the electron is given by:

$$\Phi(\vec{r}, t) = \frac{l}{4\pi\epsilon_0 c} \frac{z}{r} \frac{I(t - r/c)}{r} \tag{10}$$

5. **[4.5pt]** Compute the electric and magnetic fields emitted by the electron in the hydrogen atom when hit by the electromagnetic wave.

6. **[2pt]** Compute the Poynting flux of the radiation emitted by electron

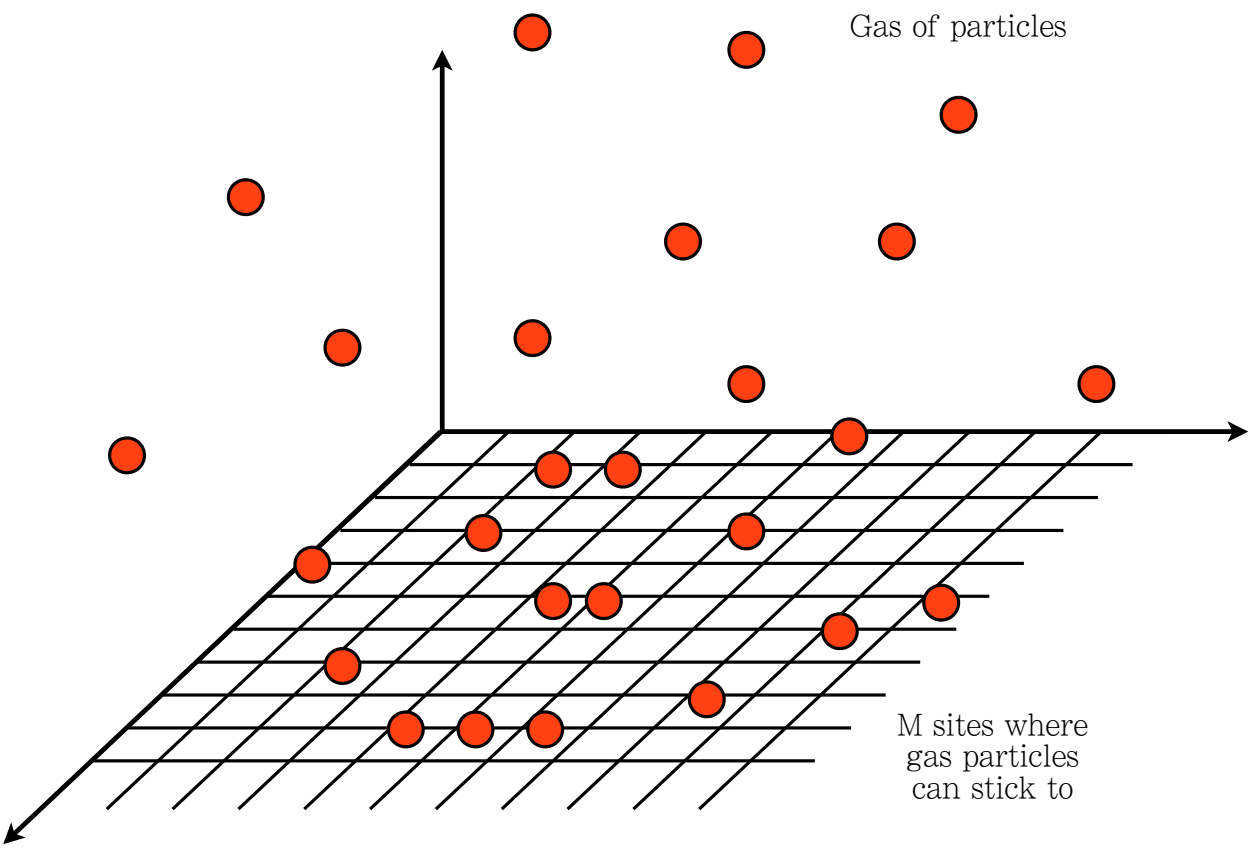
$$\vec{u} = \frac{\vec{E} \times \vec{B}}{\mu_0} \tag{11}$$

showing that the energy flux is radially outwards from the source and that it is proportional to $\omega^2 I_0^2$.

7. **[3pt]** In general the frequency of the electromagnetic wave is much smaller than the typical frequency of the atom $\omega \ll \omega_0$. Explain in this case why is the sky blue.

3 A Sticky Surface

Consider a perfect gas of N particles of mass m in a large volume V and a temperature T. This gas is in contact with a surface where some of these particles can be adsorbed onto. A simple model for this is to imagine the adsorbing surface as a lattice of M binding sites where the particles can be adsorbed onto. Each such site is either empty (in which case the associated partition function equal to 1) or occupied (with a partition function $q = e^{-\varepsilon/kT}$).



The partition function for the combined system of gas plus surface then reads

$$Z = \sum_{\text{configurations}} e^{-\frac{1}{kT}E_n} = \underbrace{\sum_{Q=0}^{M}}_{A} \underbrace{\frac{M!}{Q!(M-Q)!}}_{B} \underbrace{q^Q}_{C} \underbrace{\frac{1}{(N-Q)!}}_{D} \underbrace{V^{N-Q}\left(\frac{2\pi mkT}{h^2}\right)^{3(N-Q)/2}}_{E} \tag{12}$$

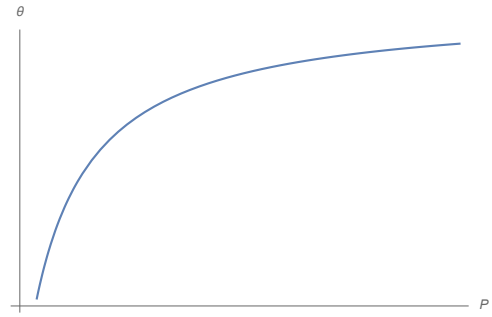


Figure 3: Langmuir Isotherm.

1. **[4pt]** Explain the origin of each of the elements in the partition function indicated by the letters $A-E$. You can regroup them if you prefer to explain the full partition function in a different way.

2. **[4pt]** Show that in the limit when the number of gas particles is much larger than the number of sites of the surface, we have

$$Z \simeq \frac{e^N}{\sqrt{2\pi N}} \left(\frac{V}{N\lambda^3}\right)^N (1+\alpha)^M \ , \qquad \lambda^2 \equiv \frac{h^2}{2\pi mkT} \tag{13}$$

   What is α?

   Hint: For large N Stirling yields $N! \simeq \sqrt{2\pi N} e^{-N} N^N$.

3. **[4pt]** Explain why in this limit $N \gg M$ the pressure $P(V,N,T) = kT\,\partial \log Z/\partial V$ is the same as that for an ideal gas – without any surface.

4. **[4pt]** Compute the average occupation fraction $\theta = \frac{\text{number of particles in the surface}}{\text{number of sites in the surface}}$ as a function of the pressure P, temperature T, adsorption partition function q and thermal wavelength λ – still in the same thermodynamic limit $N \gg M$. As a function of P you should obtain something as sketched in figure 3. This curve is called the Langmuir Isotherm.

5. **[4pt]** What happens to θ in each of the following limits. Explain the physics of each result.

   - $P \to 0$
   - $P \to \infty$
   - $T \to 0$ with positive binding energy ε.
   - $T \to 0$ with negative binding energy ε.
   - $T \to \infty$.

4 Photographing a Relativistic Sphere

Moving objects contract. Careful observations that simultaneously measure the position of different parts of an object have clearly demonstrated this contraction. In such an *observation*, the observer collects photons that left the object at the same time. These photons reach the observer at different times.

In this problem you will examine what you will see if you look at a moving object (or take a picture of a moving object). By *seeing* we mean collecting all of the photons that arrive at your eye or camera[1] at a particular time. These photons left the object at different times. The difference between *observing* and *seeing* is illustrated in figure 4.

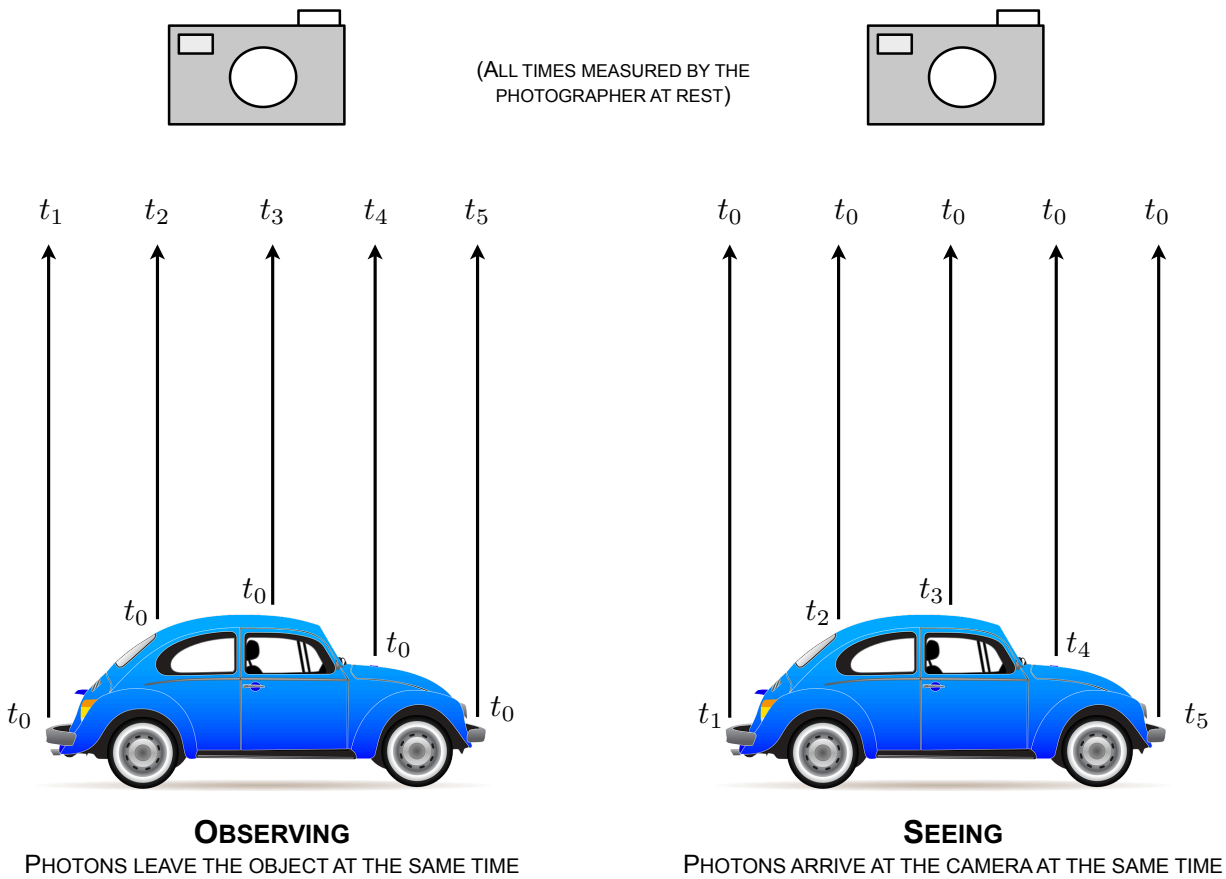


Figure 4: The difference between "observing" and "seeing" a moving object.

[1]We assume the camera has instantaneous shutter speed. To simplify things further, we assume throughout the object is far enough such that the visual solid angle is sufficiently small.

Warm-up

1. **[4pt]** *Relativistic Aberration Law*: A photon propagates at an angle α with respect to the x-axis of an inertial frame S of first observer. The first observer moves in the x' direction with speed $v = \beta c$ in the inertial frame O' of the second observer. The x and x' axes are parallel. This is demonstrated in figure 5.

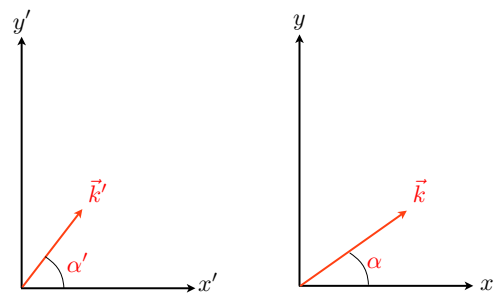


Figure 5: A photon viewed by two observers moving respect to each other.

   (a) How do the space-time coordinates transform under Lorentz transformation?

   (b) Show that the angle α' between the x' axis and the direction of propagation of the photon in frame O' is:

$$\sin\alpha' = \frac{\sqrt{1-\beta^2}\sin\alpha}{1+\beta\cos\alpha} \tag{14}$$

2. **[5pt]** *Lorentz Contracted Sphere*: A sphere of rest diameter D is traveling to the right with velocity $v = \beta c$ relative to observer O. The sphere is observed from O at an angle $\theta = \pi/2$, and is very far from the observer which we call *the photographer*, see figure 6. Write out the equation for the ellipsoid corresponding to the Lorentz contracted sphere *observed* in the photographer's frame.

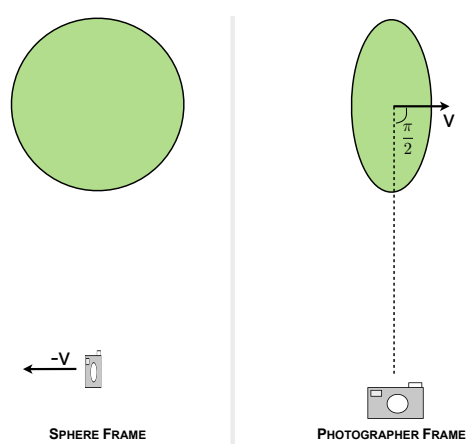


Figure 6: A sphere Lorentz contracts and is therefore *observed* as an ellipsoid.

Seeing a Moving Sphere

3. [**5pt**] A sphere of rest diameter D is traveling to the right with velocity $v = \beta c$ relative to observer O. The sphere is seen from O at an angle $\theta = \pi/2$, and is very far from the observer. In this exercise we will see that this sphere will be *seen* as a sphere by a far away observer, see figure 11. As indicated on the left of figure 7, in the photographer's frame photons leave the object perpendicular to it. Show that in the sphere frame – magnified on the right in figure 7 – these photons leave the sphere at an angle α with the value indicated in the figure.

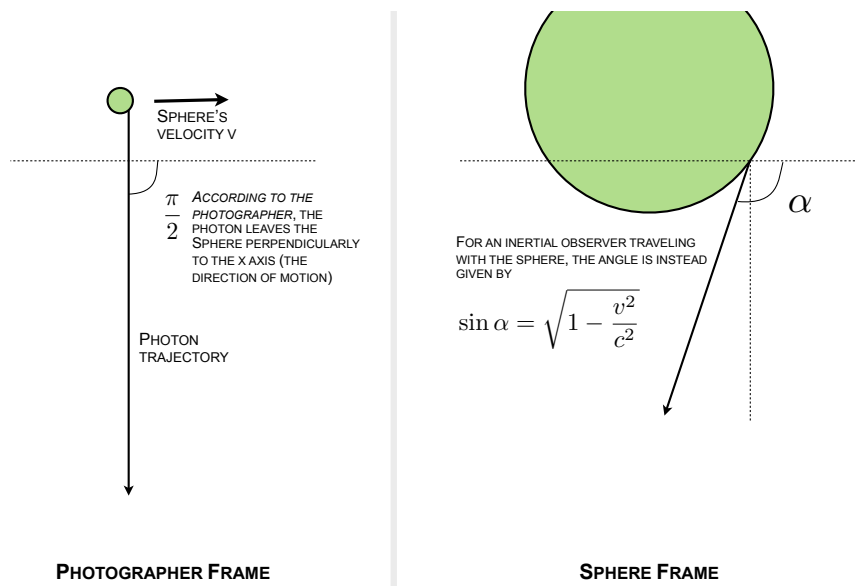


Figure 7: Photons leave the sphere at different angles (measured with respect to the direction of motion x) in the photographer and sphere frame.

4. [**5pt**] As illustrated in figure 8 using the sphere frame, the photographer will actually observe the sphere as *rotated*. What is the angle δ? What are the distances Δx_{sphere} and Δy_{sphere} in the sphere frame?

   Note that it is also possible to understand the tilting directly in the photographer's frame as depicted in figure 10 but we will not need it here.

5. [**4pt**] In the photographer frame show that the distance along the direction of motion between the farthest simultaneously visible points A and B is $\Delta x^{(2)}_{\text{photo}} = D(1 - \beta^2)$.

6. [**5pt**] Compute all other distances in figure 10. Most importantly, you should obtain that $\Delta x_{\text{photo}} = \Delta y_{\text{photo}} = D$, so that the sphere is actually *seen* as spherical! (Since in the z direction there is no contraction, the diameter is still D there also.)

   A photo of a relativistic sphere is depicted in figure 11.

To summarize: While spheres are indeed *observed* as Lorentz contracted they are still *seen* as spheres, albeit rotated ones.

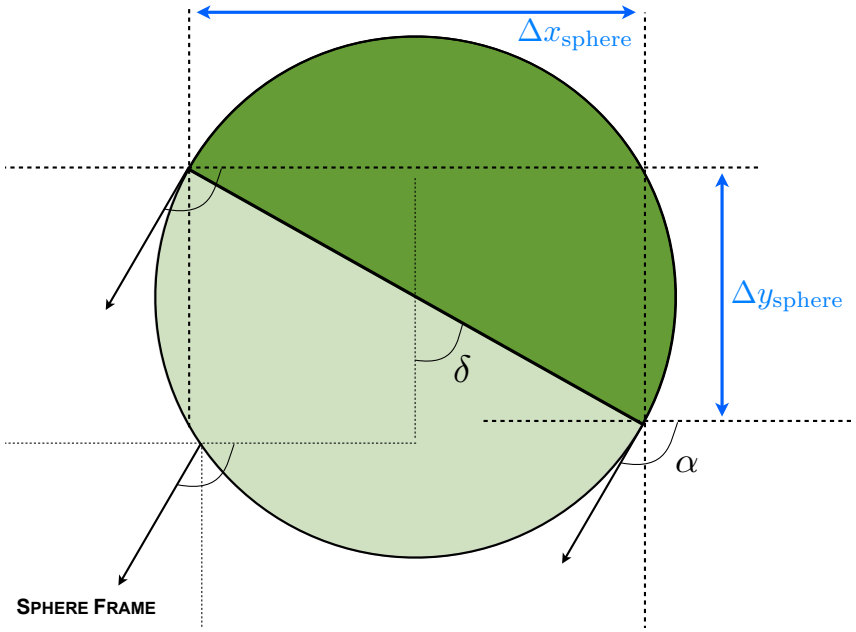


Figure 8: Since the photons leave the sphere at an angle, those in the dark region will not be able to propagate towards the photographer while those in the light region will.

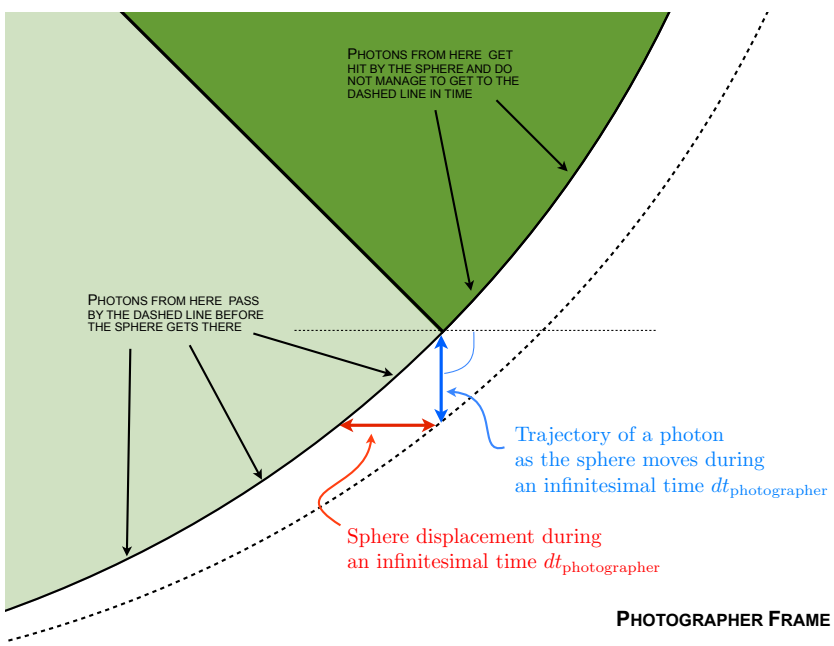


Figure 9: In the photographer's frame we see that the tilting can be explained by studying the collision of the sphere with travelling photons. (In the opposite side of the sphere the transition is explained in similar terms where now the sphere moves away from the photons instead.)

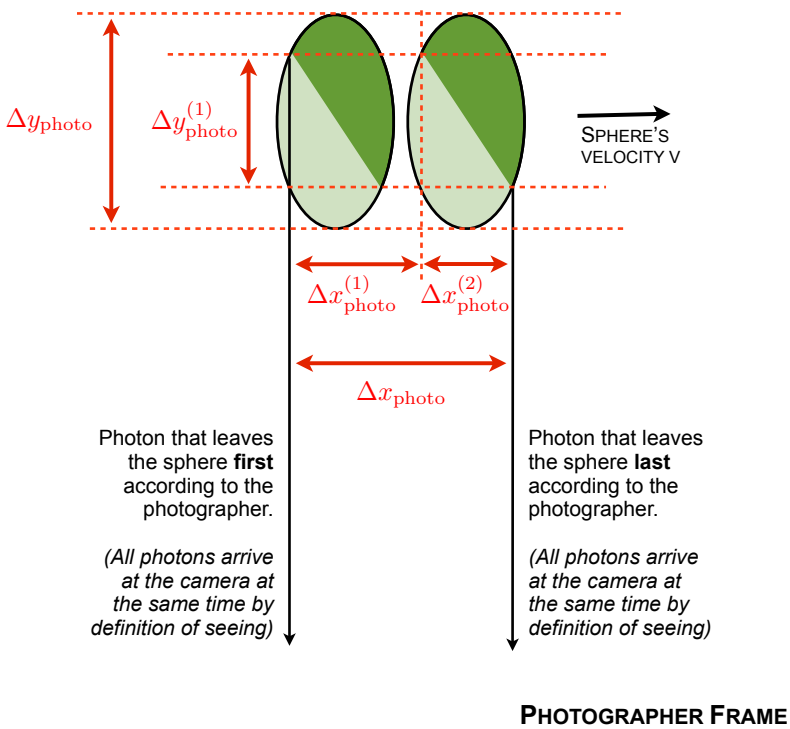


Figure 10: Various relevant distances in the photographer's frame.

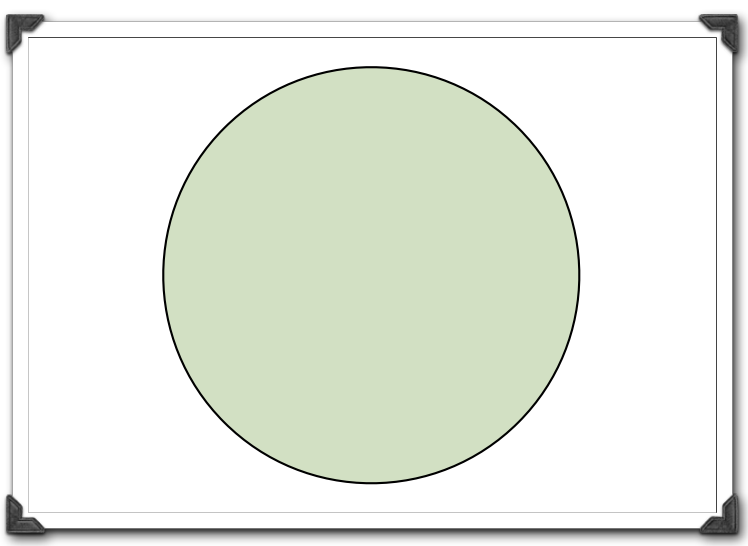


Figure 11: Photograph of a moving sphere.