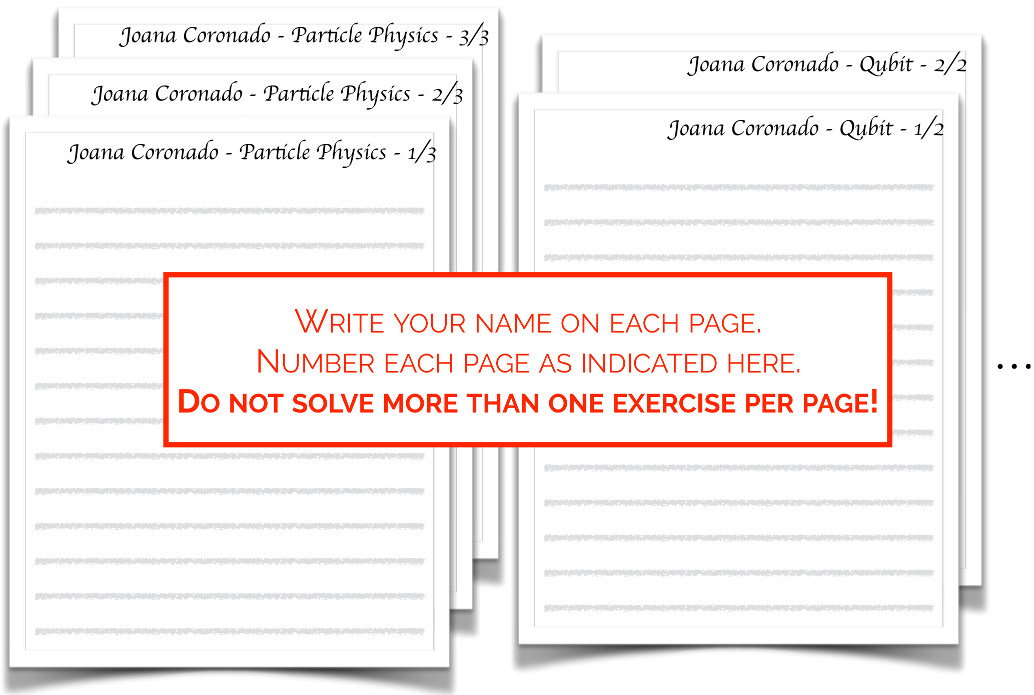
IFT-Perimeter-SAIFR

Journeys into Theoretical Physics

2016

Morning Exam

Important:

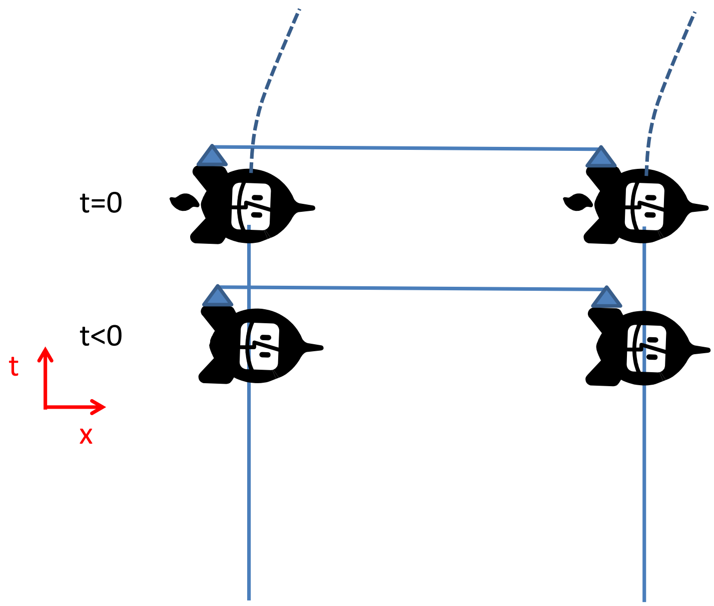


Scores:

- Problem 1: 20%
- Problem 2: 20%
- Problem 3: 20%
- Problem 4: 20%
- Problem 5: 20%

1 Bells Paradox

Consider two identical rockets initially lying at rest with respect to some given inertial congruence S. The rockets are connected by a thin cable as shown in the figure.



At $t = 0$ (according to the inertial congruence S), the rockets are simultaneously (and gently) launched. Then, they accelerate smoothly for some time along some axis according to the equations of motion

$$\text{rocket 1: } L(t) = \sqrt{(ct)^2 + 25}, \qquad 0 \leq ct \leq 10\,, \tag{1}$$
$$\text{rocket 2: } L(t) = \sqrt{(ct)^2 + 25} + 7, \qquad 0 \leq ct \leq 10\,, \tag{2}$$

where (t, L) are Cartesian coordinates (as taught in the course) defined by the original inertial congruence S. Finally at $ct = 10$ both engines are switched off, after which the rockets proceed inertially.

The cable which connects the rockets breaks down when either it is compressed by 80% (or more) or stretched by 10% (or more) of its proper length.

1. **[20pt]** Does the cable break down eventually? Explain your answer.

2 Master Equation for a Decaying Qubit

Consider a spin 1/2 particle with a Hamiltonian $H = \frac{1}{2}\vec{\Omega} \cdot \vec{\sigma}$ (where $\vec{\Omega} = \Omega \vec{e}_z$ with $\vec{e}_z$ a unit vector and $\vec{\sigma}$ the vector formed by the three Pauli operators). The qubit decays to the state $|0\rangle$ due to the interacts with an environment. This process is described by a Limbladt master equation with only one Limbladt operator: $L_1 = \gamma^{1/2}\sigma_-$, where $\sigma_- = |0\rangle\langle 1|$ is the lowering operator. .

1. **[6pt]** Write the master equation.
2. **[7pt]** Use the master equation to obtain evolution equations for the polarization vector of the qubit (whose componentts are equal to the expectation value of the three Pauli operators, i.e., $\vec{p} = Tr(\rho\vec{\sigma})$.
3. **[7pt]** Interpret the results and describe the evolution of the polarization vector in the Bloch sphere.

Hint:

The general form of the Limblad master equation is the following

$$\dot{\rho} = -i[H, \rho] + \sum_b (2L_b \rho L_b^\dagger - L_b^\dagger L_b \rho - \rho L_b^\dagger L_b), \tag{3}$$

where ρ is the density matrix, H is the Hamiltonian and L_b are some arbitrary operators.

3 Lattice Wave Packets

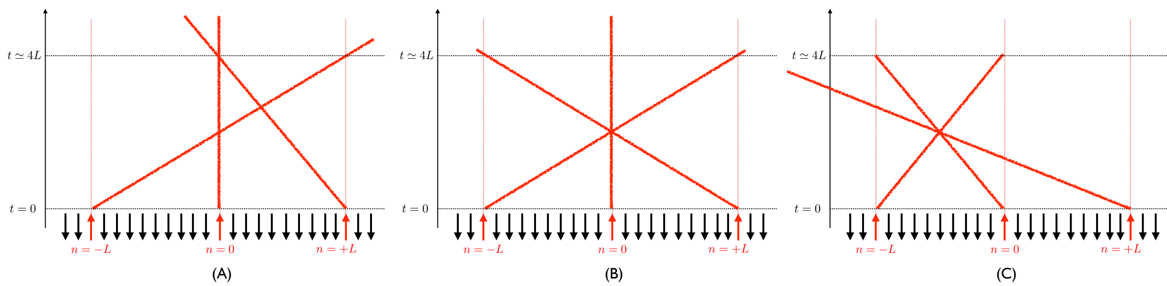


Figure 1: Three evolutions, all starting at positions $-L, 0, L$ but evolving quite differently. In the first one they meet pairwise at well separated points. In the second case they meet in the middle, at $n = 0$ while in the last case they also meet all three at the same point but this time on the left.

Consider an infinite Heisenberg spin chain with the same Hamiltonian we saw in the lectures,

$$\hat{H} = \sum_n \left(\mathbb{I}_{n,n+1} - \mathbb{P}_{n,n+1}\right) . \tag{4}$$

1. **[5pt]** Consider a wave packet at $t = 0$:

$$\psi_{\text{initial}}(n) = \int_{-\pi}^{\pi} dk\, e^{-a(k-p)^2 + ikn} \tag{5}$$

   where n is an integer denoting the spin chain site.

   (a) What is $\psi(n,t)$?, i.e. how does this wave packet evolve in time?

   (b) What is the velocity with which it moves?

   (c) Write down a translated wave packet which starts at some $n_i \neq 0$ at $t = 0$.

2. **[10pt]** Write down three initial wave packets $\psi_{initial}(n_1, n_2, n_3)$ whose time evolution would lead to the three situations in figure 1.

3. **[5pt]** Suppose now we deform a bit the Hamiltonian to something else (which would generically be non-integrable of course). For example, we could imagine adding a next-to-nearest neighbour interaction changing $\hat{H}$ into

$$\hat{H} = \sum_n \left(\left(\mathbb{I}_{n,n+1} - \mathbb{P}_{n,n+1}\right) + \beta \left(\mathbb{I}_{n,n+2} - \mathbb{P}_{n,n+2}\right)\right) \text{ with } |\beta| < 1/2 \,. \tag{6}$$

   Consider now the time evolution of the very same three wave packets you wrote down in the previous point with a new deformed Hamiltonian. Give a rough qualitative description of what you expect to get replacing figure 1.

4 Particle Physics

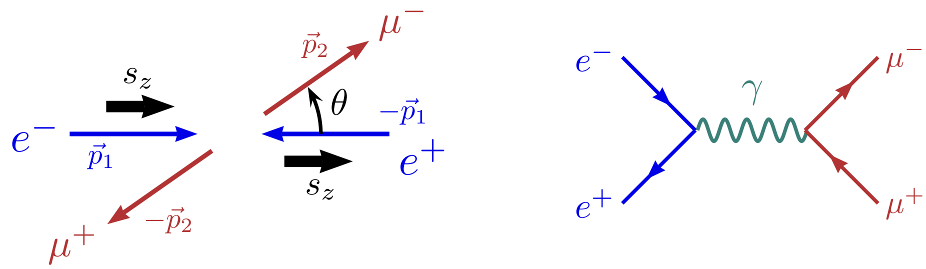


Figure 2: Left: Kinematics for $e^+e^- \to \mu^+\mu^-$ scattering. The thick arrows indicate the z-component of the spin (where, for the e^+ and e^-, the z-axis is the horizontal, i.e. beam axis). Right: Dominant Feynman diagram corresponding to photon exchange, giving rise to the previous process at energies below tens of GeV.

Consider the scattering of an electron e^- against a positron e^+ in the center of mass (CM) frame. Assume the center of mass energy to be large enough to create a $\mu^+\mu^-$ pair. The latter are, of course, produced back to back, but they can fly off along an axis different from the initial beam axis (which we call the z-axis). Define the scattering angle θ as the angle between the momentum of the incident e^- and the momentum of the scattered μ^-, as shown in the left diagram above.

In addition, assume you have prepared you beams so that they are polarized as shown: the e^- is "right-handed", (i.e. its z-spin component is in the same direction as its momentum, $s_z = +1/2$). The e^+ is instead "left-handed": its z-spin component is in the opposite direction as its momentum, which again means $s_z = +1/2$. Assume also that your detectors can measure the spin of the final muons (specifically, the projection along the momentum).

The above process can proceed through photon exchange, as shown in the right diagram above. Assuming that the CM energy is much larger than both the electron and muon masses, the corresponding differential scattering cross section as a function of scattering angle θ and the total center of mass energy E_{CM} is given by

$$\frac{d\sigma}{d\Omega}(e^+(\downarrow)\, e^-(\uparrow) \to \mu^+(\downarrow)\, \mu^-(\uparrow)) = \frac{\alpha^2}{4E_{\text{CM}}^2}\,(1+\cos\theta)^2 \ ,$$

where the arrows indicate the *projection of the spin along the direction of momentum*, and α is the fine structure constant. Physically, the previous expression gives the probability (density) that the muons be produced at an angle θ (given E_{CM}).

1. **[5pt]** Based on angular momentum conservation and your knowledge about the photon, give a physically transparent interpretation of the above angular dependence.

2. **[5pt]** From the understanding gained in the previous part, guess the angular dependence for the process

$$\frac{d\sigma}{d\Omega}(e^+(\downarrow)\, e^-(\uparrow) \to \mu^+(\uparrow)\, \mu^-(\downarrow)) \ .$$

   What about

$$\frac{d\sigma}{d\Omega}(e^+(\uparrow)\, e^-(\uparrow) \to \mu^+(\uparrow)\, \mu^-(\uparrow)) \ ?$$

3. **[5pt]** In addition to photon exchange, there is a contribution from Higgs exchange! Based on the known mass of the electron and muon ($m_e \approx 0.5$ MeV and $m_\mu \approx 100$ MeV), the mass of the Higgs (125 GeV), and what you have learned about the couplings of the Higgs boson to these particles, *estimate* how much smaller the Higgs contribution is compared to the photon exchange process above.

4. **[5pt]** Now imagine a world where the electroweak scale instead of having the value $v = 174$ GeV had a much smaller value of order the muon mass (we assume that the electron and muon masses have the same values as in our universe). For the purpose of this *gedanken* experiment, forget about anything that has to do with the QCD interactions (protons, etc. do not play a role) or other SM particles not explicitly mentioned in this problem.[1] What angular dependence do you expect for the differential scattering cross section for polarized experiments as described above? Based on this observation, can you propose a way to establish the existence of the Higgs boson in this alternate universe?

[1] To be clear, in this alternate universe you may focus on electrons, muons, photons and Higgs bosons only.

5 Scattering Amplitudes

Gluons are particles with helicity ± 1 while gravitons possess helicity ± 2. Recall that the little group of the Poincare group in four dimensions imposes severe constraints on scattering amplitudes. Explicitly, an amplitude of n particles with helicities $\{h_1, h_2, \ldots, h_n\}$ must satisfy

$$\mathcal{A}(\{t_1\lambda_1, t_1^{-1}\tilde{\lambda}_1\}, \ldots, \{t_n\lambda_n, t_n^{-1}\tilde{\lambda}_n\}) = \left(\prod_{a=1}^{n} t_a^{-2h_a}\right)\mathcal{A}(\{\lambda_1, \tilde{\lambda}_1\}, \ldots, \{\lambda_n, \tilde{\lambda}_n\}).$$

1. **[2pt]** Show that the square of an amplitude of n gluons transforms exactly as an amplitude of n gravitons under the little group.

In this problem you will explore the possibility that graviton amplitudes can be constructed from products of gluon amplitudes.

Let's start with some dimensional analysis. Let g and κ be the coupling constants for the interaction of gluons and gravitons respectively. g is dimensionless while κ has dimensions of inverse mass[2], i.e. m^{-1}.

2. **[3pt]** The coupling dependence of n-particle amplitudes of gluons and gravitons at tree-level is g^{n-2} and κ^{n-2} respectively. Therefore it is useful to write

   $$\mathcal{A}_n^{\text{gravitons}} = \delta^4(k_1+k_2+\ldots+k_n)\kappa^{n-2}A_n^{\text{gravitons}} \quad \text{and} \quad \mathcal{A}_n^{\text{gluons}} = \delta^4(k_1+\ldots+k_n)g^{n-2}A_n^{\text{gluons}}$$

   so that the quantities A_n, known as stripped amplitudes, do not depend on the coupling constants. $\mathcal{A}_n$ can then be called full amplitudes. Recall that stripped amplitudes A_n^{gluons} have dimension m^{4-n}.

   Use dimensional analysis to find the power w of Mandelstam invariants $s_{ab} = (k_a + k_b)^2$ that makes the following schematic formula dimensionally correct

   $$A_n^{\text{gravitons}} = s^w (A_n^{\text{gluons}})^2 \tag{7}$$

   (Hint: full n-particle amplitudes $\mathcal{A}_n^{\text{gravitons}}$ and $\mathcal{A}_n^{\text{gluons}}$ have the same dimension.)

In the rest of this problem we restrict our attention to amplitudes with two negative helicity particles, say i and j, and $n-2$ positive helicity ones. These amplitudes of gluons were first computed and simplified by Parke and Taylor in 1986. The result is

$$A_n^{\text{gluons}}(i^-, j^-) = \text{Tr}(T^{\mathsf{a}_1} \cdots T^{\mathsf{a}_n})\text{PT}(1, 2, \ldots, n) + \ldots \tag{8}$$

where

$$\text{PT}(1, 2, \ldots, n) := \frac{\langle i\ j\rangle^4}{\langle 1\ 2\rangle\langle 2\ 3\rangle \cdots \langle n-1\ n\rangle\langle n\ 1\rangle}$$

is known as a *partial amplitude*[3]. The $\ldots$ in (8) represent a sum over permutations of

[2] $\kappa^2 = 32\pi G_N$ in units where $G_N = 1/M_{\text{Planck}}^2$ is Newton's gravitational constant and M_{Planck} is Planck's mass.

[3] The notation PT refers to Parke-Taylor. Also, recall that $\langle a\ b\rangle := \epsilon_{\alpha\beta}\lambda_a^\alpha\lambda_b^\beta$.

the labels.

It took until 2012 for a simple formula to be constructed for gravity amplitudes! The formula uses a $n \times n$ matrix Φ defined by

$$\Phi_{ab} = \begin{cases} \frac{s_{ab}}{\langle a\ b\rangle^2}, & a \neq b; \\ -\sum_{c=1, c\neq a}^{n} \frac{s_{ac}}{\langle a\ c\rangle^2} \frac{\langle c\ x\rangle^2}{\langle a\ x\rangle^2}, & a = b. \end{cases} \tag{9}$$

Here x is a reference spinor that can be chosen arbitrarily.

This matrix has vanishing determinant. In fact, the first time one of its submatrices has non-vanishing determinant is when three rows, say a, b, c and three columns, say p, q, r, are *removed*. Let $(\Phi)^{abc}_{pqr}$ denote the $(n-3) \times (n-3)$ matrix obtained in this way. Quite surprisingly, the combination

$$\frac{\det(\Phi)^{abc}_{pqr}}{(\langle a\ b\rangle\langle b\ c\rangle\langle c\ a\rangle)(\langle p\ q\rangle\langle q\ r\rangle\langle r\ p\rangle)}$$

turns out to be independent of the choice and it is called the reduced determinant of Φ, or $\det'\Phi$. Hodges' formula for a graviton amplitude is astonishingly simple

$$A_n^{\text{graviton}}(i^-, j^-) = \langle i\ j\rangle^8 \det{}'\Phi. \tag{10}$$

3. **[2pt]** Compute $A_4^{\text{graviton}}(i^-, j^-)$ explicitly. (Hint: Choose abc and pqr so that no diagonal terms of Φ enter in $(\Phi)^{abc}_{pqr}$).

4. **[3pt]** Show that $A_4^{\text{graviton}}(i^-, j^-)$ can be written as a product of two possibly distinct 4-gluon *partial* amplitudes times Mandelstam invariants consistent with your dimensional analysis result in equation (7).

5. **[3pt]** Compute $A_5^{\text{graviton}}(i^-, j^-)$ explicitly.

6. **[7pt]** Show that each of the two terms obtained by expanding the 2×2 determinant $\det'\Phi$ can be written as the product of two 5-gluon *partial* amplitudes times Mandelstam invariants.

For some further comments with some suggestions for future research (unrelated to the exam!) see very end of the document.

Future research (Freddy's Cachazo's problem complement):

As it turns out, a direct expansion the 3×3 determinant in a six-graviton amplitude leads to terms that cannot be written directly as the product of two six-gluon partial amplitudes. In order to see more clearly what the problem is, it is convenient to introduce a graphical representation. Draw six vertices with labels $\{1, 2, \ldots, 6\}$. For each factor of $\langle a\ b \rangle$ in the denominator of a given term in the expansion of $\det' \Phi$ draw a line connecting two vertices a and b, then you should find a 4-regular graph. This means that each vertex has degree 4. A gluon-amplitude corresponds to a Hamiltonian cycle of the graph, i.e., a connected closed path in the graph that visits every vertex exactly once. Convince yourself that the 4-regular graphs you found do *not* admit a Hamiltonan decomposition, i.e., do not contain two disjoint (not sharing edges) Hamiltonian cycles. However, this situation can be fixed by using that the quantities $R(a, b, c, d) := 1/\langle a\ b \rangle \langle b\ c \rangle \langle c\ d \rangle \langle d\ a \rangle$ satisfy the following identity $R(a, b, c, d) + R(a, b, d, c) + R(a, d, b, c) = 0$. Using this identity, show that each term in the expansion of $\det' \Phi$ can written as two graphs that admit Hamiltonian decompositions and therefore can be written as the product of two six-gluon partial amplitudes. This means that a six-graviton amplitude can be written as a sum of twelve products of two six-gluon partial amplitudes. Try to generalize this to seven and more gravitons.